Math 331, Fall 2022 **Test 2 Information**

The second in-class test is scheduled to take place on Friday, November 18. It will cover Chapter 3 and Chapter 4 up to and including Section 4.5. There will be no questions specifically directed towards material from Chapters 1 and 2, but you still need to know that material as background.

Questions on the test can include definitions, statements of theorems, short and long essay questions about concepts, concrete problems, and short proofs. Any proofs on the test will be ones that I believe should be straightforward, not requiring a great deal of thought.

Some things that you should know about for the test:

differentiability of a function at a point; differentiability on an interval

partition of a closed bounded interval

upper and lower Riemann sums ($U(P, f)$ and $L(P, f)$)

infimum and supremum (inf and sup)

refinement of a partition

Riemann integrable function

the Riemann integral $\int_a^b f$

criteria for integrability

the Dirichlet function $D(x)$ — discontinuous at every point; not Riemann integrable

Taylor polynomial, $p_{n,a}$, for a function

sequences of real numbers and limits of sequences

bounded sequence

monotone sequence

Cauchy sequence

infinte series

geometric series

p-series

absolute convergence

conditional convergence

alternating series

convergence tests

sequences of functions

pointwise convergence of a sequence of functions

uniform convergence of a sequence of functions

Some definitionss:

Definition. Let f be a function defined on an open interval containing a. We say that f is **differentiable** at a if $\lim_{x\to a}\frac{f(x)-f(a)}{x-a}$ exists. We then denote the value of the limit as $f'(a)$.

Definition. A **partition** of $[a,b]$ is a sequence of points $P = \{x_o, x_1, \dots, x_n\}$ such that $a = x_o < x_1 < \cdots < x_n = b$. If P and Q are partitions, we say Q is a **refinement** of P if Q contains every point that is in P.

Definition. Let f be a bounded function on $[a,b]$, and let $P = \{x_o, x_1, \dots, x_n\}$ be a partition of $[a,b]$. For $i = 1, 2, \dots, n$, let $M_i = \sup\{f(x) \,|\, x \in [x_{i-1}, x_i]\}$ and $m_i = \inf\{f(x) \,|\, x \in [x_{i-1}, x_i]\}$. We define the **upper Riemann sum** of f relative to the partition as $U(P,f) = \sum_{i=1}^{n} M_i(x_{i-1} - x_i)$, and we define the **lower Riemann sum** of f relative to the partition as $L(P,f) = \sum_{i=1}^{n} m_i(x_{i-1} - x_i)$.

Definition. We say that a function f is **integrable** on $[a,b]$ if it is bounded on $[a,b]$ and $\sup\{L(P,f) \,|\, P \text{ is a partition of } [a,b]\}$ is equal to $\inf\{U(P,f) \,|\, P \text{ is a partition of } [a,b]\}$. In that case, their common value is denoted $\int_a^b f$ and is called the (Riemann) **integral** of f on $[a,b]$.

Definition. Let f be a function that is n times differentiable at a. The n-th degree **Taylor polynomial** for f at a is defined to be $p_{n,a}(x) = \sum_{k=0}^{n} \frac{f^{(k)}(a)}{k!}(x-a)^n$.

Definition. A sequence $\{x_n\}_{n=1}^{\infty}$ **converges** to $L \in \mathbb{R}$ if for every $\varepsilon > 0$, there is an $N \in \mathbb{N}$ such that for all $n \geq N$, $|x_n - L| < \varepsilon$. If a sequence is not convergent, then it **diverges**.

Definition. A sequence $\{x_n\}_{n=1}^{\infty}$ diverges to $+\infty$ if for every $M \in \mathbb{R}$, there is an $N \in \mathbb{N}$ such that for all $n \geq N$, $x_n > M$. A sequence $\{x_n\}_{n=1}^{\infty}$ diverges to $-\infty$ if for every $M \in \mathbb{R}$, there is an $N \in \mathbb{N}$ such that for all $n \geq N$, $x_n < M$.

Definition. A sequence $\{x_n\}_{n=1}^{\infty}$ is **non-decreasing** if for all $n > m$, $x_n \geq x_m$. It is **non-decreasing** if for all $n > m$, $x_n \leq x_m$. It is **monotone** if it is non-increasing or non-decreasing.

Definition. A sequence $\{x_n\}_{n=1}^{\infty}$ is **Cauchy** if for every $\varepsilon > 0$, there is an $N \in \mathbb{N}$ such that for all $n, m \geq N$, $|x_n - x_m| < \varepsilon$.

Definition. The n-th **partial sum** of a series $\sum_{k=1}^{\infty} a_k$ is $\sum_{k=1}^{n} a_k$.

Definition. A series **converges** to $L \in \mathbb{R}$ if the sequence of partial sums converges to L. It **diverges** if it does not converge. It **diverges to** $\pm\infty$ if the sequence of partial sums diverges to $\pm\infty$.

Definition. The series $\sum_{k=1}^{\infty} a_n$ **converges absolutely** if $\sum_{k=1}^{\infty} |a_n|$ converges. It **converges conditionally** if it converges but does not converge absolutely.

Definition. The sequence of functions $\{f_n\}_{n=1}^{\infty}$ **converges pointwise** to a function $f(x)$ on an interval I if for every $x \in I$, $\lim_{n\to\infty} f_n(x) = f(x)$.

Definition. The sequence of functions $\{f_n\}_{n=1}^{\infty}$ **converges uniformaly** to a function $f(x)$ on an interval I if for every $\varepsilon > 0$, there is an $N \in \mathbb{N}$ such that for all $n > N$ and all $x \in I$, $|f_n(x) - f(x)| < \varepsilon$,

Some theorems:

Theorem. (Differentiability implies continuity.) If a function f is differentiable at a, then f is continuous at a.

Theorem. (Properties of the derivative.) [*Insert the constant multiple rule, sum rule, product rule, quotient rule, and chain rule here.*]

Theorem. (Rolle's Theorem.) Suppose that the function f is continuous on the interval $[a, b]$ and is differentiable on (a, b), and that $f(a) = f(b) = 0$. Then there is a $c \in [a, b]$ such that $f'(c) = 0$.

Theorem. (MVT—Mean Value Theorem.) Suppose that the function f is continuous on the interval $[a, b]$ and is differentiable on (a, b). Then there is a $c \in (a, b)$ such that $f'(c) = \frac{f(b)-f(a)}{b-a}$.

Theorem. (Corollaries of the MVT.) If $f'(x) = 0$ on an interval, then f is constant on that interval. If $f'(x) = g'(x)$ on an interval, then f and g differ by a constant on that interval. If $f'(x) > 0$ on an interval, then f is increasing on that interval. If $f'(x) < 0$ on an interval, then f is decreasing on that interval. If $f'(x) \geq 0$ on an interval, then f is non-decreasing on that interval. If $f'(x) \leq 0$ on an interval, then f is non-increasing on that interval.

Theorem. Let f be a bounded function on $[a, b]$. Let P and Q be partitions of $[a, b]$ such that Q is a refinement of P. Then $L(P, f) \leq L(Q, f) \leq U(Q, f) \leq U(P, f)$.

Theorem. Let f be a bounded function on $[a, b]$, and let P and Q be any two partitions of $[a, b]$. Then $L(Q, f) \leq U(P, f)$. [This implies that the set $\{L(P, f) \,|\, P \text{ is a partition of } [a, b]\}$ is bounded above, that $\{U(P, f) \,|\, P \text{ is a partition of } [a, b]\}$ is bounded below, and also that $\sup_P(\{L(P, f)\}) \leq \inf_P(\{U(P, f)\})$.]

Theorem. Let f be a bounded function on $[a, b]$. Then f is Reimann integrable on $[a, b]$ if and only if for every $\epsilon > 0$, there is a partition P of $[a, b]$ such that $U(P, f) - L(P, f) < \epsilon$.

Theorem. If f is a non-decreasing function, or is a non-increasing function, on $[a, b]$, then f is Riemann integrable on $[a, b]$.

Theorem. If f is a continuous function on $[a, b]$, then f is Riemann integrable on $[a, b]$.

Theorem. (Linearity of the integral.) If f and g are Riemann integrable functions on $[a, b]$ and $c \in R$, then the functions cf and $f + g$ are Riemann integrable on $[a, b]$, and $\int_a^b cf = c \int_a^b f$, and $\int_a^b (f + g) = \int_a^b f + \int_a^b g$.

Theorem. (Additivity of the integral.) If f is defined on $[a, b]$ and $a < c < b$, then f is Riemann integrable on $[a, b]$ if and only if f is Riemann integrable on $[a, c]$ and f is Riemann integrable on $[c, b]$, and in that case, $\int_a^b f = \int_a^c f + \int_c^b f$. [With the usual definitions of $\int_a^b f$ for $b = a$ and for $b < a$, this formula is valid even if c is not between a and b, as long as f is integrable on an interval that contains a, b, and c.]

Theorem. Let f be an integrable function on $[a, b]$, and define $F(x) = \int_a^x f(t)\,dt$ for $x \in [a, b]$. Then F is continuous on $[a, b]$.

Theorem. (First Fundamental Theorem of Calculus.) Suppose f is an integrable function on $[a, b]$ and g is a differentiable function on $[a, b]$ satisfying $g'(x) = f(x)$ for $x \in [a, b]$. Then $\int_a^b f = g(b) - g(a)$.

Theorem. (Second Fundamental Theorem of Calculus.) Let f be a continuous function on $[a, b]$, and define $F(x) = \int_a^x f$ for $x \in [a, b]$. Then F is differentiable on $[a, b]$ and $F'(x) = f(x)$ for $x \in [a, b]$. [In fact, if we only assume that f is continuous at some point $c \in [a, b]$, then F is differentiable at c, and $F'(c) = f(c)$.]

Theorem. (Properties of limits of sequences.) [*Insert assertions about limits of sums, differences, constant multiples, products, and quotients.*]

Theorem. (Monotone Convergence Theorem.) An increasing sequence converges if and only if it is bounded (above). A decreasing sequence converges if and only if it is bounded (below). An increasing sequence that is not bounded above diveges to $+\infty$. A decreasing sequence that is not bounded below diverges to $-\infty$.

Theorem. (Cauchy Convergence Theorem.) A sequence converges if and only if it is Cauchy.

Theorem. (Linearity of Infinite Series.) If $\sum_{k=1}^{\infty} a_k$ and $\sum_{k=1}^{\infty} b_k$ are convergent series, then $\sum_{k=1}^{\infty}(a_k + b_k)$ is convergent, and $\sum_{k=1}^{\infty}(a_k + b_k) = \sum_{k=1}^{\infty} a_k + \sum_{k=1}^{\infty} b_k$. If $\sum_{k=1}^{\infty} a_k$ is a convergent series and $c \in \mathbb{R}$, then $\sum_{k=1}^{\infty} ca_k$ converges and $\sum_{k=1}^{\infty} ca_k = c\sum_{k=1}^{\infty} a_k$.

Theorem. (Geometric Series.) The geometric series $\sum_{n=0}^{\infty} ar^n$ converges to $\frac{a}{1-r}$ if $|r| < 1$. If $|r| \geq 1$ (and $a \neq 0$), then the series diverges.

Theorem. (p-series). The p-series $\sum_{n=1}^{\infty} \frac{1}{n^p}$ converges if $p > 1$ and diverges to $+\infty$ if $p \leq 1$.

Theorem. (n-th Term Test.) If the sequence $\{a_k\}_{k=1}^{\infty}$ does not converge to 0, then the series $\sum_{k=1}^{\infty} a_n$ diverges.

Theorem. (Ratio Test.) Suppose that $\lim\limits_{k\to\infty} \frac{|a_{k+1}|}{|a_k|} = L$ (where L can be a number or $+\infty$). If $L < 1$, then the series $\sum_{k=1}^{\infty} a_k$ converges absolutely. If $L > 1$, then the series diverges. (The case $L = 1$ gives no information about the series.)

Theorem. If the series $\sum_{k=1}^{\infty} a_k$ converges absolutely, then it converges.

Theorem. (Alternating Series Test.) Let $\{a_k\}_{k=1}^{\infty}$ be a sequence of positive terms. If the sequence is decreasing and $\lim\limits_{k\to\infty} a_k = 0$, then the alternating series $\sum_{k=1}^{\infty}(-1)^{k+1}a_k$ converges.

Theorem. Suppose that $\{f_n\}_{n=1}^{\infty}$ is a sequence of continuous functions that converges uniformly to a function $f(x)$ on an interval I. Then f is continuous on I.

Theorem. Suppose that $\{f_n\}_{n=1}^{\infty}$ is a sequence of functions that are integrable on the interval $[a, b]$ and that the sequence converges uniformly to a function $f(x)$ on $[a, b]$. Then f is integrable on $[a, b]$, and $\lim\limits_{n\to\infty} \left(\int_a^b f_n(x)\, dx\right) = \int_a^b f(x)\, dx$.