This test is due by the start of class on Monday, October 3. **You should do any six out of the seven problems on the test. Do not do all seven; if you do all seven, your answer for problem 7 will be ignored.**

You should do this test on your own, using only your textbook, class notes, and previous work in the course as reference. You should not work with other students, you should not use the Internet or other references, and you should not consult anyone except the professor for the course. You can ask questions about the test in class and by email. I might give some hints and clarifications, but I am unlikely to give extensive, detailed help. Be sure to show all of your work! If you are not able to complete a problem, you should turn in work showing whatever progress you have made on it, for partial credit. **Note: There will be no "rewrites" for this test.**

You can submit your answers in LaTeX on overleaf.com, if you want to do that, but you are welcome to write up your answers neatly by hand.

Although some problems are more difficult than others, and some are fairly easy, all six problems will count equally.

Problem 1. State a careful definition of $\lim\limits_{x \to a^+} f(x) = +\infty$. Then use the definition to prove directly that $\lim\limits_{x \to 0^+} \dfrac{1}{x} = +\infty$.

Problem 2. Let X and Y be non-empty, bounded subsets of $\mathbb{R}$. Suppose that for every $x \in X$ and for every $y \in Y$, $x < y$. Prove that $lub(X) \leq glb(Y)$. Is it always true that $lub(X) < glb(Y)$? (Prove or give a counterexample!)

Problem 3. Let A and B be subsets of $\mathbb{R}$. Suppose that x is an accumulation point of the set $A \cup B$. Show that x is an accumulation point of A or x is an accumulation point of B (or both). (Hint: Try a proof by contradiction.)

Problem 4. Let f and g be functions. Then we can define a new function $\max(f, g)$ whose value at x is given by $\max(f(x), g(x))$.

(a) Show that for any numbers a and b, $\max(a, b) = \frac{1}{2}\big(|a - b| + a + b\big)$. (Hint: Consider two cases.)

(b) Now, suppose that f and g are continuous on an interval I. Show that the function $\max(f, g)$ is also continuous on I. Be clear about what continuity rules or theorems you use.

Problem 5. Let S be a subset of $\mathbb{R}$. Recall that S is said to be *dense* in $\mathbb{R}$ if for any open interval (a, b), the intersection of S with the set (a, b) is not empty. (That is, there is at least one $s \in S$ such that $a < s < b$.) Prove that S is dense in $\mathbb{R}$ if and only if every point of $\mathbb{R}$ is an accumulation point of S.

Problem 6. Let $f(x)$ be a continuous function on a closed, bounded interval $[a, b]$. In class, we used uniform continuity of f to show that f is bounded above. However, it is possible to prove that directly using the Heine-Borel Theorem. Follow this outline to prove that there is a number M such that $f(x) \leq M$ for all $x \in [a, b]$:

- Show that for any $z \in [a, b]$, there is a $\delta_z > 0$ and a number M_z such that $f(x) \leq M_z$ for all $x \in (z - \delta_z, z + \delta_z)$. (This is an easy consequence of continuity. Just let $\varepsilon = 1$ in the definition of continuity at z, and get $f(x) < f(z) + 1$ for all x near enough to z.)
- Define an open cover of $[a, b]$ consisting of the intervals $(z - \delta_z, z + \delta_z)$, for all $z \in [a, b]$. (State why it is a cover.)
- Apply the Heine-Borel Theroem, and finish the proof.

Problem 7. Suppose that $f(x)$ and $g(x)$ are uniformly continuous on the interval I (which is not necessarily closed or bounded). Show directly from the definition of uniform continuity that $f(x) + g(x)$ is uniformly continuous on I.