This homework is due on Wednesday, November 2.
Note that problems 1 and 2 are not proofs and are not eligible for revision.

Problem 1. Let f be the polynomial $f(x) = 2 - 5x^2 + 3x^3 - x^4$. Use Taylor's Theorem to write f as a polynomial in powers of $(x+1)$. (That is, find the Taylor polynomial, $p_{4,-1}(x)$, of degree 4 at -1 for f.)

Problem 2. Find the general Taylor polynomial at 0, $p_{n,0}(x)$, for the function $\ln(x+1)$.

Problem 3 (from 4.2.14 from the textbook)**.** We have shown that the n^{th} Taylor polynomial for e^x at 0 is $p_{n,0}(x) = \sum_{k=1}^{n} \frac{1}{n!}x^n$. Show that e is irrational by using proof by contradiction. Suppose, for the sake of contradiction, that $e = \frac{p}{q}$ for some integers p and q.

(a) Use the Lagrange form of the remainder term from Taylor's Theorem to show that there is a $c \in [0,1]$ such that $\frac{p}{q} - \left(\frac{1}{0!} + \frac{1}{1!} + \cdots + \frac{1}{n!}\right) = \frac{e^c}{(n+1)!}$.

(b) Multiply both sides of the equation in (a) by $n!$, and show that left side of the resulting equation is an integer when $n \geq q$.

(c) Show that the right side of the equation that you got in part (b) is not an integer when $n > e$. Conclude that e is irrational.

Problem 4. Suppose that f is a function defined for all $x \geq 1$ and that $\lim_{x\to+\infty} f(x) = L$. Define a sequence $\{a_n\}_{n=1}^{\infty}$ by $a_n = f(n)$ for all $n \in \mathbb{N}$. Prove that $\lim_{n\to\infty} a_n = L$.

Problem 5. Prove that if $\{x_n\}_{n=1}^{\infty}$ is an increasing sequence that is not bounded above, then $\lim_{n\to\infty} x_n = +\infty$.

Problem 6 (From Textbook problem 4.2.5)**.** Let $\{a_n\}_{n=1}^{\infty}$ be defined inductively as follows:

$$a_1 = 1, \qquad a_n = 1 + \frac{a_{n-1}}{4} \quad \text{for } n > 1$$

(a) Show by induction that a_n is bounded above by $4/3$.

(b) Show that $\{a_n\}_{n=1}^{\infty}$ is convergent by showing that it is increasing.

(c) Show that $\lim_{n\to\infty} a_n = 4/3$. [Hint: Use the fact that $\lim_{n\to\infty} a_n = \lim_{n\to\infty} a_{n+1}$ and the recursive definition of a_n.]

Problem 7. Suppose that $\{a_n\}_{n=1}^{\infty}$ and $\{b_n\}_{n=1}^{\infty}$ are sequences, and $\{a_n\}_{n=1}^{\infty}$ is convergent with $\lim_{n\to\infty} a_n = L$. Suppose in addition that $\lim_{n\to\infty} |a_n - b_n| = 0$. Show that $\{b_n\}_{n=1}^{\infty}$ is convergent and $\lim_{n\to\infty} b_n = L$.