This homework on Metric Spaces is due Friday, October 7

Problem 1. Let (M, d) be a metric space, and let A be a subset of M. Prove that A is open if and only if A is equal to a union of open balls.

Problem 2. Consider the metric space $(\mathbb{R}, d)$, where d is the usual metric on $\mathbb{R}$ For each $n = 1, 2, 3, \dots$, let $\mathcal{O}_n$ be the open set $\mathcal{O}_n = \left(1, 1 + \frac{1}{n}\right)$. Show that $\{\mathcal{O}_n \mid n = 1, 2, \dots\}$ is an infinite collection of open sets whose intersection is not open. And find an infinite collection of closed subsets of $(\mathbb{R}, d)$ whose union is not closed.

Problem 3. Let X be any non-empty, bounded subset of $\mathbb{R}$, and let λ be the least upper bound of X. Show that $\lambda \in \overline{X}$. That is, the least upper bound of any set is an element of the closure of that set. [Hint: Use the definition of closure of X as the set of all points of X plus all accumulation points of X, and use Problem 1 from Homework 3.]

Problem 4. Let (A, σ), (B, τ), and (C, η) be metric spaces. Let $f\colon A \to B$ and $g\colon B \to C$. Suppose that f and g are continuous functions. Prove that their composition, $g \circ f$, is a continuous function.

Problem 5. Let $\{x_n\}_{n=1}^{\infty}$ be a convergent sequence in a metric space. Show that its limit is unique. That is, prove the following statement: if $\{x_n\}_{n=1}^{\infty}$ converges to y and $\{x_n\}_{n=1}^{\infty}$ converges to z, then $y = z$.