This homework is due by 4:00 PM on Monday, September 26

Problem 1. Suppose that $f(x)$ is defined and bounded on an open interval containing 0, except possibly at 0 itself. (That is, there is a number B such that $|f(x)| < B$ for all x in that interval, except possibly $x = 0$.) Show that $\lim_{x\to 0} xf(x) = 0$. [Hint: The product rule does not apply here. Use the Squeeze Theorem and the fact that $|x|$ is a continuous function.]

Problem 2. If $f(x)$ is a continuous function, then we know that $|f(x)|$ is also continuous, since it is a composition of continuous functions. Give a counterexample to show that the converse does not hold. That is, find a function $f(x)$ such that $|f(x)|$ is continuous, but $f(x)$ is not continuous.

Problem 3 (Textbook problem 2.5.7)**.** Suppose that f is continuous at a and that $f(a) > 0$. Prove that there is a $\delta > 0$ such that $f(x) > 0$ for all x in the interval $(a - \delta, a + \delta)$.

Problem 4 (Textbook problem 2.4.10)**.** Prove: If $\lim_{x\to a^+} f(x) = L$ and if $c(x)$ is a function such that $a < c(x) < x$ for all x in some interval (a, b), then $\lim_{x\to a^+} f(c(x)) = L$. [Hint: This is confusing but actually easy.]

Problem 5. Let f be a continuous function on the interval $[a, b]$, and suppose that $f(x) \in \mathbb{Q}$ for all $x \in [a, b]$. Show that f is constant on $[a, b]$. [Hint: Use the Intermediate Value Theorem.]

Problem 6 (Textbook problem 2.6.7b)**.** Show that $p(x) = x^4 - x^3 + x^2 + x - 1$ has at least two roots in the interval $[-1, 1]$.

Problem 7. Show that any linear function $f(x) = mx + b$ is uniformly continuous on $\mathbb{R}$.

Problem 8. Let $f(x) = \frac{1}{x}$.

(a) Show that for any $c > 0$, $f(x)$ is uniformly continuous on $[c, \infty)$,

(b) Show that $f(x)$ is not uniformly continuous on $(0, \infty)$.

Problem 9 (Textbook problem 2.6.12a)**.** We say that a function f satisfies a **Lipschitz condition** if there is a positive real number M such that for all $x, y \in \mathbb{R}$, $|f(x) - f(y)| < M|x - y|$. Show that if f satisfies a Lipschitz condition, then f is uniformly continuous on $(-\infty, \infty)$.