This homework can be turned in until 3:00 PM on Thursday, September 15.

Problem 1 (Textbook problem 1.4.12a)**.** Suppose that λ is the least upper bound of some set S, and that λ is *not* in S. Prove that λ is an accumulation point of S. [Hint: For any $\varepsilon > 0$, there is a point $s \in S$ such that $\lambda - \varepsilon < s < \lambda$. Now use the definition of accumulation point to finish the proof.]

Problem 2 (Textbook problems 1.4.9 and 1.4.10)**. (a)** Prove lemma 1.4.5: If x is an accumulation point of a set S and if $\varepsilon > 0$, then there is an infinite number of points of S within distance ε of S. That is, $(x - \varepsilon, x + \varepsilon) \cap X$ is infinite. [Hint: Given $\varepsilon > 0$, suppose that there is only a finite number of points, $s_1, s_2, \ldots, s_k$, of S within ε of x, but not equal to x. Let $\varepsilon' = \min(|s_1 - x|, |s_2 - x|, \ldots, |s_k - x|)$. Now, show that no $s \in S$ satisfies $0 < |s - x| < \varepsilon'$.] **(b)** Deduce that if S is a **finite** subset of $\mathbb{R}$, then S has no accumulation points. [This is trivially a corollary of the lemma.]

Problem 3. Prove directly, using the (epsilon-delta) definition of limits, that $\lim\limits_{x \to 5} \frac{2x+4}{7} = 2$.

Problem 4. Show directly, without using the product rule for limits, that $\lim\limits_{x \to 3} x^3 = 27$. (Note that $a^3 - b^3 = (a - b)(a^2 + ab + b^2)$.)

Problem 5. Suppose that $\lim\limits_{x \to a} f(x) = L$ and $c \in \mathbb{R}$. Prove directly, using the definition of limit, that $\lim\limits_{x \to a} cf(x) = cL$. [Be careful: $c = 0$ is a special case.]

Problem 6 (Textbook problem 2.2.9)**.** Suppose that $f(x) \leq 0$ for all x in some open interval containing a, except possibly at a. Suppose that $\lim\limits_{x \to a} f(x) = L$. Show that $L \leq 0$. [Hint: Assume instead that $L > 0$. Let $\varepsilon = L/2$ and derive a contradiction.] (Remark: A similar proof shows that if $f(x) \geq 0$ for all x near a, then $\lim\limits_{x \to a} f(x) \geq 0$, if the limit exists.)

Problem 7. This problem gives an alternative proof of the product rule.

(a) Suppose $\lim\limits_{x \to a} f(x) = L$. Show directly from the definition of limit (without using the product rule) that $\lim\limits_{x \to a} f(x)^2 = L^2$.

(b) Verify algebraically, by expanding the right-hand side, that $ab = \frac{1}{4}\big((a+b)^2 - (a-b)^2\big)$.

(c) Let's say that the sum, difference, and constant multiple rules for limits have already been proved, in addition to parts (a) and (b) of this problem. Using all that (and **not** the definition of derivative), prove the product rule for limits.