**Math 331** **Homework 2**

*This homework is due by the end of the day on Friday, September 7*
*Problems are mostly on Sections 1.3 and 1.4, not including Bolzano-Weirstrass.*

**Problem 1.** Prove using only the defintion of real numbers as Dedekind cuts and the definitions of $+$ and $<$ in terms of Dedekind cuts: If $\alpha, \beta, \delta \in \mathbb{R}$ and $\alpha < \beta$, then $\alpha+\delta < \beta+\delta$.

**Problem 2** (From Problem 1.3.7 in the textbook)**.** Suppose that $(\mathbb{F}, +, \cdot)$ is a field, and $S \subseteq \mathbb{F}$. We say that $S$ is a subfield of $\mathbb{F}$ if it is a field under the same addition and multiplication as $\mathbb{F}$. To show that $S$ is a subfield of $\mathbb{F}$, it is enough to show that $0 \in S$, $1 \in S$, and $S$ is closed under addition, multiplication, taking additive inverses, and taking multiplicative inverses..

Let $\mathbb{Q}[\sqrt{2}] = \{r + s\sqrt{2} \,|\, r, s \in \mathbb{Q}\}$. Show that $\mathbb{Q}[\sqrt{2}]$ is a subfield of $\mathbb{R}$. (Note: Remember that $r$ and $s$ can be zero in $r + s\sqrt{2}$.)

**Problem 3** (Problem 1.3.11 from the textbook)**.** Let $(F, +, \cdot)$ be an ordered field. Use the definition of $x < y$ and the order axioms to prove the transitive property of $<$. That is, show that for any $a, b, c \in \mathbb{F}$, if $a < b$ and $b < c$, then $a < c$. [Note: Since $\mathbb{F}$ is not necessarily $\mathbb{R}$, you can't use common facts that you know about $\mathbb{R}$. You can only use the actual definition and axioms.]

**Problem 4. (a)** Let $\mathcal{O}_1, \mathcal{O}_2 \ldots, \mathcal{O}_k$ be some finite number of open subsets of $\mathbb{R}$. Prove that their intersection, $\bigcap_{i=1}^{k} \mathcal{O}_i$, is open. (Hint: Use the characterization of open that involves $\varepsilon > 0$. Start by taking arbitrary $x \in \bigcap_{i=1}^{k} \mathcal{O}_i$.)

**(b)** Show that the intersection of an infinite number of open sets is not necessarily open by finding $\bigcap_{n=1}^{\infty} \left(-1 - \frac{1}{n}, 1 + \frac{1}{n}\right)$. (Justify your answer!)

**Problem 5.** Consider the **unbounded** closed interval $[0, \infty)$. Find an open cover of this interval that has no finite subcover. (This problem shows that the hypothesis that the interval is bounded cannot be removed from the Heine-Borel Theorem. Use a simple example, but justify your answer!)

**Problem 6** (Problem 1.4.3 from the textbook)**.** Suppose that $\{\mathcal{O}_\alpha \,|\, \alpha \in A\}$ is an open cover of the interval $[0, 1)$. Suppose furthermore that $1 \in \bigcup_{\alpha \in A} \mathcal{O}_\alpha$. Prove that there is finite subcover of $[0, 1)$ from $\{\mathcal{O}_\alpha \,|\, \alpha \in A\}$. [This question tests your understanding of the proof of the Heine-Borel Theorem.]

**Problem 7.** Let $f(x)$ be a real-valued function that is defined on an interval $I$. We say that $f$ is bounded above on $I$ if there is a number $M$ such that $f(x) < M$ for all $x \in I$.

Suppose that $f(x)$ is defined on the bounded, closed interval $[a, b]$. Suppose that for every $x \in [a, b]$, there is an $\varepsilon > 0$ such that $f$ is bounded above on the interval $(x - \varepsilon, x + \varepsilon)$. Use the Heine-Borel theorem to prove that $f$ is bounded above on $[a, b]$. (Hint: Compare this to an example about functions that was done in class.)