Problem 1. Prove using only the definition of real numbers as Dedekind cuts and the definitions of $+$ and $<$ in terms of Dedekind cuts: If $\alpha, \beta, \delta \in \mathbb{R}$ and $\alpha < \beta$, then $\alpha+\delta < \beta+\delta$.

Suppose $\alpha, \beta, \delta \in \mathbb{R}$ and $\alpha < \beta$. To show $\alpha + \delta < \beta + \delta$, we must show $\alpha + \delta \subset \beta + \delta$. Let $p \in \alpha + \delta$. We must show $p \in \beta + \delta$. By definition of addition of Dedekind cuts, $p = a + c$ where $a \in \alpha$ and $c \in \delta$. Since $\alpha < \beta$ and $a \in \alpha$, then $a \in \beta$. Since $a \in \beta$ and $c \in \delta$, then $a + c \in \beta + \delta$. Since $p = a + c$, we have shown $p \in \beta + \delta$.

My answer is, in fact, incomplete. To show $\alpha + \delta < \beta + \delta$, we must show that $\alpha + \delta$ is a **proper** subset of $\beta + \delta$. I have shown $\alpha + \delta \subset \beta + \delta$, but it remains to show $\alpha + \delta \neq \beta + \delta$.

Problem 2 (From Problem 1.3.7 in the textbook). [From Problem 1.3.7 in the textbook] Suppose that $(\mathbb{F}, +, \cdot)$ is a field, and $S \subseteq \mathbb{F}$. We say that S is a subfield of $\mathbb{F}$ if it is a field under the same addition and multiplication as $\mathbb{F}$. To show that S is a subfield of $\mathbb{F}$, it is enough to show that $0 \in S$, $1 \in S$, and S is closed under addition, multiplication, taking additive inverses, and taking multiplicative inverses..

Let $\mathbb{Q}[\sqrt{2}] = \{r + s\sqrt{2} \,|\, r, s \in \mathbb{Q}\}$. Show that $\mathbb{Q}[\sqrt{2}]$ is a subfield of $\mathbb{R}$. (Note: Remember that r and s can be zero in $r + s\sqrt{2}$.)

Let $S = \mathbb{Q}[\sqrt{2}]$.

1. $0 \in S$, since it can be written as $0 = 0 + 0\sqrt{2}$, and $1 \in S$ because $1 = 1 + 0\sqrt{2}$.

2. Let $a, b \in S$. Then $a = r + s\sqrt{2}$ and $b = p + q\sqrt{2}$ for some $r, s, p, q \in \mathbb{Q}$. Then $a + b = (r + s\sqrt{2}) + (p + q\sqrt{2}) = (r + p) + (s + q)\sqrt{2}$, and $r + p$ and $s + q$ are in $\mathbb{Q}$ because $\mathbb{Q}$ is closed under addition. So, $a + b \in S$. Thus, S is closed under addition.

3. With a and b as in item 2, $ab = (r + s\sqrt{2})(p + q\sqrt{2}) = (rp + rq\sqrt{2} + ps\sqrt{2} + qs(\sqrt{2})^2 = (rp + 2qs) + (rq + ps)\sqrt{2}$, which is in S because $\mathbb{Q}$ is closed under multiplication and addition. Thus, S is closed under multiplication.

4. Let $a \in S$, where $a = r + s\sqrt{2}$. Then $-a = (-r) + (-s)\sqrt{2}$, which is in S. So, the additive inverse of an element of S is in S.

5. Finally, let $r + s\sqrt{2} \in S$ be a non-zero element of S. Saying it is non-zero means at least one of r or s is non-zero. Note that $r^2 - 2s^2 \neq 0$. (Suppose $r^2 - 2s^2 = 0$. Then $r^2 = 2s^2$. Since one of r and s is non-zero and $r^2 = 2s^2$, they both must be non-zero. But then we have $2 = \frac{r^2}{s^2}$, and $\sqrt{2} = \frac{|r|}{|s|}$, which is impossible because $\sqrt{2}$ is not rational.) We have $\big(r + s\sqrt{2}\big)\big(\frac{r - s\sqrt{2}}{r^2 - 2s^2}\big) = \frac{r^2 - 2s^2}{r^2 - 2s^2} = 1$. So the multiplicative inverse of $r + s\sqrt{2}$ is $\frac{r - s\sqrt{2}}{r^2 - 2s^2}$, which can be written as $\frac{r}{r^2 - 2s^2} + \frac{-s}{r^2 - 2s^2}\sqrt{2}$, which is in S. Thus, the multiplicative inverse of any non-zero element of S is in S.

Problem 3 (Problem 1.3.11 from the textbook). Let $(F, +, \cdot)$ be an ordered field. Use the definition of $x < y$ and the order axioms to prove the transitive property of $<$. That is, show that for any $a, b, c \in \mathbb{F}$, if $a < b$ and $b < c$, then $a < c$. [Note: Since $\mathbb{F}$ is not necessarily $\mathbb{R}$, you can't use common facts that you know about $\mathbb{R}$. You can only use the actual definition and axioms.]

Let $a, b, c \in \mathbb{F}$. Suppose $a < b$ and $b < c$. Let P be the set of positive elements of $\mathbb{F}$. Since $a < b$, then by definition, $b - a \in P$. Similarly, $c - b \in P$. Since P is closed under addition, $(b-a)+(c-b) \in P$. Using properties of addition and additive inverse, this becomes $c - a \in P$. And then, by definition of "less than," $a < c$.

Problem 4. (a) Let $\mathcal{O}_1, \mathcal{O}_2 \dots, \mathcal{O}_k$ be some finite number of open subsets of $\mathbb{R}$. Prove that their intersection, $\bigcap_{i=1}^{k} \mathcal{O}_i$, is open. (Hint: Use the characterization of open that involves $\varepsilon > 0$. Start by taking arbitrary $x \in \bigcap_{i=1}^{k} \mathcal{O}_i$.)

(b) Show that the intersection of an infinite number of open sets is not necessarily open by finding $\bigcap_{n=1}^{\infty} \left(-1 - \frac{1}{n}, 1 + \frac{1}{n} \right)$. (Justify your answer!)

(a) A set G is open if for all $x \in G$, there is an $\varepsilon > 0$ such that $(x - \varepsilon, x + \varepsilon) \subseteq G$. Let $x \in \bigcap_{i=1}^{k} \mathcal{O}_i$. We must find some $\varepsilon > 0$ such that $(x - \varepsilon, x + \varepsilon) \subseteq \bigcap_{i=1}^{k} \mathcal{O}_i$. By definition of intersection, $x \in \mathcal{O}_i$ for every i. Since $\mathcal{O}_i$ is open, then by definition, we can find $\varepsilon_i > 0$ such that $(x - \varepsilon_i, x + \varepsilon_i) \subseteq \mathcal{O}_i$. Let $\varepsilon = \min(\varepsilon_1, \varepsilon_2, \dots, \varepsilon_k)$. Then $\varepsilon > 0$ and for each i, $(x - \varepsilon, x + \varepsilon) \subseteq (x - \varepsilon_i, x + \varepsilon_i) \subseteq \mathcal{O}_i$. Since this is true for $i = 1, 2, \dots, k$, we see that $(x - \varepsilon, x + \varepsilon) \subseteq \bigcap_{i=1}^{k} \mathcal{O}_i$.

(b) The intervals $\left(-1 - \frac{1}{n}, 1 + \frac{1}{n} \right)$ are open sets, but $\bigcap_{n=1}^{\infty} \left(-1 - \frac{1}{n}, 1 + \frac{1}{n} \right) = [-1, 1]$, which is not open, so the intersection of infinitely many open sets does not have to be open. To see that the intersection is $[-1, 1]$, note that $[-1, 1] \subset \left(-1 - \frac{1}{n}, 1 + \frac{1}{n} \right)$ for all n, so $[-1, 1]$ is a subset of their intersection. On the other hand, if $x > 1$, then $x < 1 + \frac{1}{n}$ for some $n \in \mathbb{N}$, so x is not in the intersection. That is, no number bigger than 1 is in the intersection. Similarly, no number less than -1 is in the intersection. So the intersection is exactly $[-1, 1]$.

Problem 5. Consider the **unbounded** closed interval $[0, \infty)$. Find an open cover of this interval that has no finite subcover. (This problem shows that the hypothesis that the interval is bounded cannot be removed from the Heine-Borel Theorem. Use a simple example, but justify your answer!)

One possible answer$\{(-1, n) \,|\, n = 0, 1, 2, \dots\}$. Consider any finite subset, $\{(-1, n_i) \,|\, i = 1, 2, \dots, k\}$. Let $N = 1 + \max(n_1, n_2, \dots, n_k)$ Then N is not in any of the sets $(-1, n_i)$, so those sets do not cover all of $[0, \infty]$. That is, there is no finite subset of the open cover that is itself a cover.

Another possible answer is $\{(n-1, n+1) \,|\, n = 0, 1, 2, \dots\}$. Note that each of the intervals in this set covers exactly one integer. A subset containing k open intervals from the open cover will cover only k integers, so does not cover all of $[0, \infty)$.

Problem 6 (Problem 1.4.3 from the textbook)**.** Suppose that $\{\mathcal{O}_\alpha \,|\, \alpha \in A\}$ is an open cover of the interval $[0, 1)$. Suppose furthermore that $1 \in \bigcup_{\alpha \in A} \mathcal{O}_\alpha$. Prove that there is finite subcover of $[0, 1)$ from $\{\mathcal{O}_\alpha \,|\, \alpha \in A\}$. [This question tests your understanding of the proof of the Heine-Borel Theorem.]

Since $1 \in \bigcup_{\alpha \in A} \mathcal{O}$, there is a $\beta \in A$ such that $1 \in \mathcal{O}_\beta$. Since $\mathcal{O}_\beta$ is open, there is an $\varepsilon > 0$ such that $(1 - \varepsilon, 1 + \varepsilon) \subseteq \mathcal{O}_\beta$. Choose any $b \in (0, 1)$ such that $1 - \varepsilon < b < 1$. The bounded, closed interval $[0, b]$ is a subset of $[1, 0)$, and so is covered by $\{\mathcal{O}_\alpha \,|\, \alpha \in A\}$. By the Heine-Borel Theorem, there is a finite subcover, $\{\mathcal{O}_{\alpha_1}, \mathcal{O}_{\alpha_2}, \dots, \mathcal{O}_{\alpha_k}\}$, of $[0, b]$. But $\mathcal{O}_\beta$ covers $[b, 1]$, so $\{\mathcal{O}_\beta, \mathcal{O}_{\alpha_1}, \mathcal{O}_{\alpha_2}, \dots, \mathcal{O}_{\alpha_k}\}$ is a finite subcover for all of $[0, 1)$.

[For an even easier proof, note that since $[0,1) \subseteq \bigcup_{\alpha \in A} \mathcal{O}_\alpha\}$ and $1 \in \bigcup_{\alpha \in A} \mathcal{O}_\alpha\}$, then in fact $\{\mathcal{O}_\alpha \,|\, \alpha \in A\}$ is an open cover of the closed, bounded interval $[0,1]$. By the Heine-Borel theorem, there is a finite subcover of $[0,1]$, which is automatically a subcover for $[0,1)$ because $[0,1) \subseteq [0,1]$.]

Problem 7. Let $f(x)$ be a real-valued function that is defined on an interval I. We say that f is bounded above on I if there is a number M such that $f(x) < M$ for all $x \in I$.

Suppose that $f(x)$ is defined on the bounded, closed interval $[a,b]$. Suppose that for every $x \in [a,b]$, there is an $\varepsilon > 0$ such that f is bounded above on the interval $(x-\varepsilon, x+\varepsilon)$. Use the Heine-Borel theorem to prove that f is bounded above on $[a,b]$. (Hint: Compare this to an example about functions that was done in class.)

For each $x \in [a,b]$, let $\varepsilon_x > 0$ such that f is bounded above on the inteval $(x-\varepsilon_x, x+\varepsilon_x)$, and let M_x be an upper bound for x on that interval. That is, $f(t) < M_x$ for all t in the interval $(x-\varepsilon_x, x+\varepsilon_x)$. The collection of open intervals $\{(x-\varepsilon_x, x+\varepsilon_x) \,|\, x \in [a,b]\}$ is an open cover of $[a,b]$. By the Heine-Borel Theorem, there is a finite subcover, $\{(x-\varepsilon_{x_i}, x+\varepsilon_{x_i}) \,|\, i = 1,2,\dots,k\}$. Let $M = \max(M_{x_1}, M_{x_2}, \dots, M_{x_k})$. We claim that M is an upper bound for f on all of $[a,b]$. Let $t \in [a,b]$. We must show $f(t) < M$. But there is a j such that $t \in (x-\varepsilon_{x_j}, x+\varepsilon_{x_j})$, and it follows that $f(t) < M_{x_j} \leq M$.