This homework is due by the end of the day on Wednesday, August 31.
It covers Sections 1.1 and 1.2 from the textbook.

Problem 1 (Exercise 1.1.12)**.** Prove that if a is irrational, then $\sqrt{a}$ is also irrational.

Problem 2 (Exercises 1.1.14)**.** Show that $\sqrt{3}+\sqrt{2}$ is irrational as follows: First, show that if $\sqrt{3}+\sqrt{2}$ is rational then so is $\sqrt{3}-\sqrt{2}$. (Hint: Consider their product.) Second, show that $\sqrt{3}+\sqrt{2}$ and $\sqrt{3}-\sqrt{2}$ cannot both be rational. (Hint: Consider their sum.)

Problem 3. Determine whether each set is bounded above and if so find its least upper bound. Remember to briefly explain your answers. For D and E, you will need to quote some well-know facts about the relevant infinite series.

$$A = \{1 - \tfrac{1}{n} \,|\, n \in \mathbb{N}\}$$

$$B = \{1 + \tfrac{1}{n} \,|\, n \in \mathbb{N}\}$$

$$C = [2, 9)$$

$$D = \{1, 1 + \tfrac{1}{2}, 1 + \tfrac{1}{2} + \tfrac{1}{4}, 1 + \tfrac{1}{2} + \tfrac{1}{4} + \tfrac{1}{8}, 1 + \tfrac{1}{2} + \tfrac{1}{4} + \tfrac{1}{8} + \tfrac{1}{16}, \dots\}$$

$$E = \{1, 1 + \tfrac{1}{2}, 1 + \tfrac{1}{2} + \tfrac{1}{3}, 1 + \tfrac{1}{2} + \tfrac{1}{3} + \tfrac{1}{4}, 1 + \tfrac{1}{2} + \tfrac{1}{3} + \tfrac{1}{4} + \tfrac{1}{5}, \dots\}$$

Problem 4 (From exercise 1.2.6)**.** Let A and B be arbitrary non-empty, bounded-above sets of real numbers. Define $C = \{a + b \,|\, a \in A \text{ and } b \in B\}$. [That is, C contains contains all sums made up of one number from A and one number from B.]

(a) Suppose that μ_1 is an upper bound for A and μ_2 is an upper bound for B. Let $\mu = \mu_1 + \mu_2$. Show that is an upper bound for C.

(b) Now suppose that λ_1 is the least upper bound for A and λ_2 is the least upper bound for B. Let $\lambda = \lambda_1 + \lambda_2$. Show that λ is the least upper bound for C. (Hint: Use the last theorem in the third reading guide: Let $\varepsilon > 0$. Explain why there is an $a_o \in A$ such that $a_o > \lambda_1 - \frac{\varepsilon}{2}$ and a $b_o \in B$ such that $b_o > \lambda_2 - \frac{\varepsilon}{2}$. Use this to show $a_o + b_o > \lambda - \varepsilon$, and conclude that λ is the least upper bound for C.)

Problem 5 (From exercise 1.2.4)**.** Consider two sequences of real numbers $A = \{a_1, a_2, a_3, \dots\}$ and $B = \{b_1, b_2, b_3, \dots\}$, which are bounded above. Let C be the set $C = \{a_1 + b_1, a_2 + b_2, a_3 + b_3, \dots\}$. [Compare this to the previous problem, where C contains only the sums of all elements of A with all elements of B; the C in this problem contains only sums of corresponding elements from the two sequences.]

(a) Suppose that μ_1 is an upper bound for A and μ_2 is an upper bound for B. Show that $\mu_1 + \mu_2$ is an upper bound for C.

(b) Now suppose that λ_1 is the least upper bound for A and λ_2 is the least upper bound for B. Give an example to show that $\lambda_1 + \lambda_2$ is not necessarily the least upper bound of C. [Hint: Take part (c) into account as you look for an example!]

(c) Show that if A and B are *non-decreasing* sequences, then λ is in fact the least upper bound of C. (Non-decreasing here means $a_1 \leq a_2 \leq a_3 \leq \cdots$ and $b_1 \leq b_2 \leq b_3 \leq \cdots$.)

Problem 6. The last theorem in the third reading guide is about least upper bounds. State the corresponding theorem for greatest lower bounds. You do not have to prove the theorem.

Problem 7 (Exercises 1.2.17 aamd 1.2.18)**.**

(a) Prove that the intersection of two Dedekind cuts is again a Dedekind cut.

(b) Show that the intersection of an infinite number of Dedekind cuts is not necessarily a Dedekind cut, even if the intersection is non-empty, by using the following example: For $n \in \mathbb{N}$, let S_n be the Dedekind cut corresponding to the number $\frac{1}{n}$. You need to show that $\bigcap_{n=1}^{\infty} S_n$ is not a Dedekind cut.