Consider the following linear system of four equations in four variables:

$$\begin{array}{cccccc} & y & +z & +w & = & 2 \\ 2x & +y & -2z & -2w & = & -3 \\ -x & -3y & & +w & = & 4 \\ x & -y & -z & & = & 3 \end{array}$$

The augmented matrix for this system is:

$$\left(\begin{array}{cccc|c} 0 & 1 & 1 & 1 & 2 \\ 2 & 1 & -2 & -2 & -3 \\ -1 & -3 & 0 & 1 & 4 \\ 1 & -1 & -1 & 0 & 3 \end{array}\right)$$

Do a row reduction on this matrix:

$$\left(\begin{array}{cccc|c} 0 & 1 & 1 & 1 & 2 \\ 2 & 1 & -2 & -2 & -3 \\ -1 & -3 & 0 & 1 & 4 \\ 1 & -1 & -1 & 0 & 3 \end{array}\right) \xrightarrow{\rho 1 \leftrightarrow \rho 2} \left(\begin{array}{cccc|c} -1 & -3 & 0 & 1 & 4 \\ 2 & 1 & -2 & -2 & -3 \\ 0 & 1 & 1 & 1 & 2 \\ 1 & -1 & -1 & 0 & 3 \end{array}\right)$$

$$\xrightarrow[-\rho_1+\rho_4]{2\rho_1+\rho_2} \left(\begin{array}{cccc|c} -1 & -3 & 0 & 1 & 4 \\ 0 & -5 & -2 & 0 & 5 \\ 0 & 1 & 1 & 1 & 2 \\ 0 & -4 & -1 & 1 & 7 \end{array}\right)$$

$$\xrightarrow{\rho_2 \leftrightarrow \rho_3} \left(\begin{array}{cccc|c} -1 & -3 & 0 & 1 & 4 \\ 0 & 1 & 1 & 1 & 2 \\ 0 & -5 & -2 & 0 & 5 \\ 0 & -4 & -1 & 1 & 7 \end{array}\right)$$

$$\xrightarrow[4\rho_2+\rho 4]{5\rho_2+\rho_3} \left(\begin{array}{cccc|c} -1 & -3 & 0 & 1 & 4 \\ 0 & 1 & 1 & 1 & 2 \\ 0 & 0 & 3 & 5 & 15 \\ 0 & 0 & 3 & 5 & 15 \end{array}\right)$$

$$\xrightarrow{-\rho_3+\rho_4} \left(\begin{array}{cccc|c} -1 & -3 & 0 & 1 & 4 \\ 0 & 1 & 1 & 1 & 2 \\ 0 & 0 & 3 & 5 & 15 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

This is in echelon form. The leading variables are x, y, and z. The variable w is free. We can solve this equation easily, using w as a parameter. First, use the third equation to solve for z in terms of w:

$$\begin{aligned} 3z + 5w &= 15 \\ 3z &= 15 - 5w \\ z &= 5 - \frac{5}{3}w \end{aligned}$$

Then use the second equation to solve for y, again using w as a parameter:

$$\begin{aligned} y + z + 2 &= 2 \\ y &= 2 - z - w \\ &= 2 - \left(5 - \frac{5}{3}w\right) - w \\ &= -3 + \frac{2}{3}w \end{aligned}$$

Finally, use the first equation to solve for x:

$$\begin{aligned} -x - 3y + w &= 4 \\ x + 3y - w &= -4 \\ x &= -4 - 3y + w \\ &= -4 - 3\left(-3 + \frac{2}{3}w\right) + w \\ &= 5 - w \end{aligned}$$

We can write the solution as

$$\begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix} = \begin{pmatrix} 5 - w \\ -3 + \frac{2}{3}w \\ 5 - \frac{5}{3}w \\ w \end{pmatrix}$$

or as a set,

$$\left\{ \begin{pmatrix} 5 - w \\ -3 + \frac{2}{3}w \\ 5 - \frac{5}{3}w \\ w \end{pmatrix} : w \in \mathbb{R} \right\}$$

or, using vector multiplication and addition, as

$$\left\{ \begin{pmatrix} 5 \\ -3 \\ 5 \\ 0 \end{pmatrix} + w \cdot \begin{pmatrix} -1 \\ \frac{2}{3} \\ \frac{5}{3} \\ 1 \end{pmatrix} \ : \ w \in \mathbb{R} \right\}$$