This homework is due by 11:59 PM on Tuesday, November 10

Problem 1. Remember that the change of basis matrix for bases B and D of the same vector space V is defined to be $Rep_{B,D}(id)$, where $id\colon V \to V$ is the identity map.

Suppose that B and D are bases for $\mathscr{P}_2$, where $D = \langle x+x^2, 1-2x+x^2, 2-x\rangle$. Find the basis B if the change of basis matrix is $\mathrm{Rep}_{B,D}(id) = \begin{pmatrix} -1 & 2 & 0 \\ 2 & 3 & -1 \\ 0 & 1 & 4 \end{pmatrix}$. (This problem is very easy!)

Problem 2. Consider the bases B and D for $\mathbb{R}^3$, as given here:

$$B = \left\langle \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}, \begin{pmatrix} 0 \\ -3 \\ 2 \end{pmatrix}, \begin{pmatrix} -2 \\ 4 \\ -3 \end{pmatrix} \right\rangle \qquad D = \left\langle \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}, \begin{pmatrix} 0 \\ -1 \\ 0 \end{pmatrix} \right\rangle$$

(a) Find the change of basis matrix $\mathrm{Rep}_{B,D}(id)$.

(b) Check that $\mathrm{Rep}_B \begin{pmatrix} -1 \\ 0 \\ 4 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}$. (You're not asked to find the representation, just check it.)

(c) The change of basis matrix must satisfy $\mathrm{Rep}_{B,D}(id) \cdot \mathrm{Rep}_B(\vec{v}) = \mathrm{Rep}_D(\vec{v})$. Verify that in fact

$$\mathrm{Rep}_{B,D}(id) \cdot \mathrm{Rep}_B \begin{pmatrix} -1 \\ 0 \\ 4 \end{pmatrix} = \mathrm{Rep}_D \begin{pmatrix} -1 \\ 0 \\ 4 \end{pmatrix}$$

Problem 3. Find an affine transformation $f\colon \mathbb{R}^2 \to \mathbb{R}^2$ such that

$$f\begin{pmatrix} 0 \\ 0 \end{pmatrix} = \begin{pmatrix} -2 \\ 3 \end{pmatrix} \qquad f\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \qquad f\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 2 \\ 5 \end{pmatrix}$$

Hint: It's easy to find the translation part of the affine map! Recall that f can be represented in the form $f\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} ax+by+e \\ cx+dy+f \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} e \\ f \end{pmatrix}$.

Problem 4. The cross product of two vectors $\vec{v}, \vec{w} \in \mathbb{R}^3$ is a vector, $\vec{v} \times \vec{w}$, that is orthogonal to both $\vec{v}$ and $\vec{w}$. The cross product can be computed as the "formal determinant"

$$\begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} \times \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix} = \begin{vmatrix} \vec{e}_1 & \vec{e}_2 & \vec{e}_3 \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$$

Find $\begin{pmatrix} 3 \\ -1 \\ 2 \end{pmatrix} \times \begin{pmatrix} 1 \\ 4 \\ -2 \end{pmatrix}$ by writing out the formal determinant $\begin{vmatrix} \vec{e}_1 & \vec{e}_2 & \vec{e}_3 \\ 3 & -1 & 2 \\ 1 & 4 & -2 \end{vmatrix}$, using the formula for the determinant of a 3×3 matrix, and show that the result is, in fact, orthogonal to both vectors.

Problem 5. We looked at a formula for computing the determinant of a 3×3 matrix. That formula can be derived using Laplace's expansion for the determinant. Apply Laplace's expansion to the general 3×3 determinant

$$\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

to derive the formula for the determinant of a 3×3 matrix. (You only need to apply Laplace's expansion for the first step of the computation, not for the resulting 2×2 matrices.)

Problem 6. Compute each of the following determinants. Some of them are very easy, using properties of the determinant, and you should check for that before doing a complex computation. For any problem where you use a property of the determinant to find the answer, state the property that you use.

(a) $\begin{vmatrix} 3 & 5 \\ 2 & -1 \end{vmatrix}$ **(b)** $\begin{vmatrix} 5 & 7 & 4 \\ 0 & 2 & 3 \\ 0 & 0 & 3 \end{vmatrix}$ **(c)** $\begin{vmatrix} 1 & 3 & -2 \\ 2 & 1 & 1 \\ 3 & 2 & 4 \end{vmatrix}$

(d) $\begin{vmatrix} 0 & 0 & 3 \\ 0 & 2 & -3 \\ 4 & 5 & -1 \end{vmatrix}$ **(e)** $\begin{vmatrix} 1 & 2 & 4 & -1 \\ 3 & 5 & -3 & 7 \\ 1 & 2 & 4 & -1 \\ 6 & 2 & 3 & 7 \end{vmatrix}$ **(f)** $\begin{vmatrix} 1 & 2 & 3 & -1 \\ 0 & -1 & 2 & 3 \\ 2 & 5 & 6 & 1 \\ -1 & 1 & 1 & 3 \end{vmatrix}$