Problem 1. Two people solve a linear system of equations in two variables and they get the following solution sets, where each set represents a line in $\mathbb{R}^2$:

$$A = \left\{ \begin{pmatrix} 3 \\ 2 \end{pmatrix} + a \cdot \begin{pmatrix} 1 \\ 3 \end{pmatrix} \;:\; a \in \mathbb{R} \right\} \qquad B = \left\{ \begin{pmatrix} 1 \\ -4 \end{pmatrix} + a \cdot \begin{pmatrix} 2 \\ 6 \end{pmatrix} \;:\; a \in \mathbb{R} \right\}$$

Can they both be correct? Explain why the two lines are actually the same line. First check that the point $(3, 2)$ is on both lines. Then explain why the two lines point in the same direction. And explain in words why all this shows that the two lines are the same.

Note that the solutions sets are given as set of vectors, but we usually think of lines as sets of points, so in this problem, we are thinking of a vector $\begin{pmatrix} x \\ y \end{pmatrix}$ as being the same as the point (x, y). To see that $(3, 2)$ is on the first line, just let $a = 0$ in the first set. To see $(3, 2)$ is on the second line, we need to have $3 = 1 + 2a$ and $2 = -4 + 6a$. From the first equation, a must be 1, and $a = 1$ also works for the second equation. So $(3, 2)$ is on the second line (because $\begin{pmatrix} 3 \\ 2 \end{pmatrix}$ can be written as $\begin{pmatrix} 1 \\ -4 \end{pmatrix} + a \cdot \begin{pmatrix} 2 \\ 6 \end{pmatrix}$ with $a = 1$).

The first line points in the direction $\begin{pmatrix} 1 \\ 3 \end{pmatrix}$ while the second points in the direction $\begin{pmatrix} 2 \\ 6 \end{pmatrix}$. But because $\begin{pmatrix} 2 \\ 6 \end{pmatrix} = 2 \begin{pmatrix} 1 \\ 3 \end{pmatrix}$, one vector is a scalar multiple of the other, which means that they point in the same direction.

Since the two lines have a point in common and point in the same direction, they are in fact the same line.

Problem 2. Suppose two planes in $\mathbb{R}^3$ are given by the linear equations $x + y + z = 1$ and $Ax + By + Cz = D$. The intersection of the two planes can be empty, or it can be a line, or the planes could be identical. For each case, what has to be true about the constants A, B, C, and D in the second equation? Explain! (Hint: The intersection is the set of solutions to a system of two linear equations, and that set can be determined by putting the system into echelon form.)

The system is put into echelon form by applying the row operation $-A\rho_1 + \rho_2$, giving

$$\begin{array}{cccc} x & +y & +z & = & 1 \\ & (B-A)y & +(C-A)z & = & (D-A) \end{array}$$

If either $B - A = 0$ and $C - A = 0$, then the second row is of the form $0 = (D - A)$. If $D - A$ is also 0, then the row is of the form $0 = 0$, and there are two free variables. In that case, the solution set is a plane. If $D - A \neq 0$, then the row is of the form $0 = k$ for $k \neq 0$, so there is no solution, which means that the planes have empty intersection. So, the intersection is a plane when B, C, and D are all equal to A, that is when all four coefficients are equal. There is no intersection when A, B, and C are equal, but D is different. Finally, in the case when A, B, and C are not all equal, then at least one of $B - A$ or $C - A$ is not zero, there is one free variable, and the solution set is a line.

Problem 3. Let $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_{n-1}$ be $n - 1$ vectors in $\mathbb{R}^n$. Prove that there is a non-zero vector $\vec{x}$ in $\mathbb{R}^n$ that is orthogonal to $\vec{v}_i$ for all $i = 1, 2, \ldots, n - 1$. (Hint: Think about linear equations! Write the condition as a linear system, and note that it is a homogeneous system.)

We want to find a vector $\vec{x}$ such that $\vec{x} \cdot \vec{v}_i = 0$ for $i = 1, 2, \ldots, n - 1$. Write the vector $\vec{x}$ as $\vec{x} = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}$ and write $\vec{v}_i$ as $\vec{v}_i = \begin{pmatrix} v_{i1} \\ v_{i2} \\ \vdots \\ v_{in} \end{pmatrix}$ for $i = 1, 2, \ldots, n-1$. Then $\vec{x} \cdot \vec{v}_i = 0$ becomes the equation $v_{i1}x_1 + v_{i2}x_2 + \cdots v_{in}x_n = 0$. Here, the v_{ij} are constants from the known vectors $\vec{v}_i$. So, we have a homogeneous system of $n-1$ linear equations in the n variables $x_1, x_2, \ldots, x_n$. Since there are more variables than equations, there must be at least one free variable, which means that there is an infinite number of solutions. In particular, there is a solution that is different from $\vec{x} = \vec{0}$. That solution gives us a non-zero vector that satisfies all of the equations, and the equations say that the vector is orthogonal to all of the vectors $\vec{v}_i$.

Problem 4. Let $\vec{v}_1 = \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix}$, $\vec{v}_2 = \begin{pmatrix} 3 \\ 0 \\ 1 \end{pmatrix}$, and $\vec{v}_3 = \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix}$. Write the vector $\begin{pmatrix} 0 \\ 5 \\ -1 \end{pmatrix}$ as a linear combination of $\vec{v}_1$, $\vec{v}_2$, and $\vec{v}_3$. To find the coefficients in the linear combination, set up a system of linear equations, and then solve that system.

We want to find scalars x, y, and z such that $x\vec{v}_1 + y\vec{v}_2 + z\vec{v}_3$ is the given vector. That is,

$$x \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix} + y \begin{pmatrix} 3 \\ 0 \\ 1 \end{pmatrix} + z \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix} = \begin{pmatrix} 0 \\ 5 \\ -1 \end{pmatrix}$$

or

$$\begin{array}{cccccc} x & +3y & +z & = & 0 \\ 2x & & +z & = & 5 \\ -x & +y & +2z & = & -1 \end{array}$$

We need to solve this system for x, y, and z. We begin by putting it into echelon form:

$$\begin{array}{rrrrr} x & +3y & +z & = & 0 \\ 2x & & +z & = & 5 \\ -x & +y & +2z & = & -1 \end{array} \xrightarrow[\rho_1+\rho_3]{-2\rho_1+\rho_2} \begin{array}{rrrrr} x & +3y & +z & = & 0 \\ & -6y & -z & = & 5 \\ & 4y & +3z & = & -1 \end{array}$$

$$\xrightarrow{\frac{2}{3}\rho_2+\rho_3} \begin{array}{rrrrr} x & +3y & +z & = & 0 \\ & -6y & -z & = & 5 \\ & & +\frac{7}{3}z & = & \frac{7}{3} \end{array}$$

We can then solve for x, y, and z:

$$\begin{aligned} \tfrac{7}{3}z &= \tfrac{7}{3} \\ z &= 1 \end{aligned} \qquad \begin{aligned} -6y - z &= 5 \\ -6y &= 5 + z \\ -6y &= 5 + 1 \\ y &= -1 \end{aligned} \qquad \begin{aligned} x + 3y + z &= 0 \\ x &= -3y - z \\ x &= -3(-1) - 1 \\ x &= 2 \end{aligned}$$

So, the linear combination that we want is $2\vec{v}_1 - \vec{v}_2 + \vec{v}_3$.

Problem 5. Apply Gauss's method to put each matrix into echelon form. Based on your answer, state whether the matrix is singular or non-singular.

(a) $\begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}$ **(b)** $\begin{pmatrix} -1 & 0 & 1 & 0 \\ 3 & 2 & -2 & 4 \\ 2 & 1 & 0 & -1 \\ 1 & 1 & 0 & 1 \end{pmatrix}$

(a) $\begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix} \xrightarrow{-\rho_1+\rho_2} \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$.

The matrix is nonsingular because there are no free variables, so the corresponding homogeneous system has just one solution.

(b)

$$\begin{pmatrix} -1 & 0 & 1 & 0 \\ 3 & 2 & -2 & 4 \\ 2 & 1 & 0 & -1 \\ 1 & 1 & 0 & 1 \end{pmatrix} \xrightarrow[\rho_1 + \rho_4]{\substack{3\rho_1 + \rho_2 \\ 1\rho_2 + \rho_3}} \begin{pmatrix} -1 & 0 & 1 & 0 \\ 0 & 2 & 1 & 4 \\ 0 & 1 & 2 & -1 \\ 0 & 1 & 1 & 1 \end{pmatrix}$$

$$\xrightarrow{\rho_2 \leftrightarrow \rho_3} \begin{pmatrix} -1 & 0 & 1 & 0 \\ 0 & 1 & 2 & -1 \\ 0 & 2 & 1 & 4 \\ 0 & 1 & 1 & 1 \end{pmatrix}$$

$$\xrightarrow[-\rho_2 + \rho_4]{-2\rho_2 + \rho_3} \begin{pmatrix} -1 & 0 & 1 & 0 \\ 0 & 1 & 2 & -1 \\ 0 & 0 & -3 & 6 \\ 0 & 0 & 1 & 2 \end{pmatrix}$$

$$\xrightarrow{-\frac{1}{3}\rho_3 + \rho_4} \begin{pmatrix} -1 & 0 & 1 & 0 \\ 0 & 1 & 2 & -1 \\ 0 & 0 & -3 & 6 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

The matrix is singular because there is a free variable, so the corresponding homogeneous system has infinitely many solutions.