Problem 1. A few problems on complex numbers...

(a) Recall that for a complex number $z = a + bi$, $|z|$ is defined to be $|z| = \sqrt{a^2 + b^2}$. Verify that for two complex numbers z and w, $|zw| = |z| \cdot |w|$.

(b) Suppose that $w_0 = a + bi$ is some complex number. Recall that the conjugate of w_0 is defines as $\overline{w_0} = a - bi$. Let $p(z)$ be the polynomial $p(z) = (z - w_0)(z - \overline{w_0})$. Verify that when $p(z)$ is written in standard form as $p(z) = c_0 + c_1 z + c_2 z^2$, all of the coefficients c_0, c_1, and c_2 are real.

(c) Use the identity $e^{i\theta} = \cos(\theta) + i\sin(\theta)$ and the fact that $(e^{i\theta})^2 = e^{2i\theta}$ to derive the usual double angle formulas: $\cos(2\theta) = \cos^2(\theta) - \sin^2(\theta)$ and $\sin(2\theta) = 2\sin(\theta)\cos(\theta)$.

Answer:

(a) Write $z = a + bi$ and $w = c + di$. Then $zw = (ac - bd) + (ad + bc)i$, and

$$\begin{aligned}
|zw| &= \sqrt{(ac-bd)^2 + (ad+bc)^2} \\
&= \sqrt{(a^2c^2 - 2abcd + b^2d^2) + (a^2d^2 + 2abcd + b^2c^2)} \\
&= \sqrt{a^2c^2 + b^2d^2 + a^2d^2 + b^2c^2} \\
&= \sqrt{a^2c^2 + a^2d^2 + b^2c^2 + b^2d^2} \\
&= \sqrt{a^2)(c^2 + d^2) + b^2(c^2 + d^2)} \\
&= \sqrt{(a^2 + b^2)(c^2 + d^2)} \\
&= \sqrt{a^2 + b^2} \cdot \sqrt{c^2 + d^2} \\
&= |z| \cdot |w|
\end{aligned}$$

(b) $(z - w_0)(z - \overline{w_0}) = z^2 - (w_0 + \overline{w_0})z + w_0\overline{w_0}$. Now, $w_0 + \overline{w_0} = (a + bi) + (a - bi) = 2a$, which is real, and $w_0\overline{w_0} = (a + b_i)(a - b_i) = a^2 - (bi)^2 = a^2 + b^2$, which is also real.

(c) We have $e^{2i\theta} = \cos(2\theta) + i\sin(2\theta)$. But also,

$$\begin{aligned}
e^{2i\theta} &= (e^{i\theta})^2 \\
&= (\cos(\theta) + i\sin(\theta))^2 \\
&= (\cos^2(\theta) - \sin(\theta)) + (2\cos(\theta)\sin(\theta))i
\end{aligned}$$

Equating the real and imaginary parts of the two formulas for $e^{2i\theta}$ gives the formulas stated in the problem.

Problem 2. Find all the eigenvalues, real or complex, of the following matrices. (Note that one of these is really easy.)

(a) $\begin{pmatrix} 1 & 2 \\ -2 & 1 \end{pmatrix}$ **(b)** $\begin{pmatrix} 3+i & 0 & 5 \\ 0 & 2 & 7 \\ 0 & 0 & -i \end{pmatrix}$ **(c)** $\begin{pmatrix} 5 & 0 & 0 \\ -2 & 3 & 6 \\ 0 & 1 & -2 \end{pmatrix}$

Answer:

(a) $\begin{vmatrix} 1-x & 2 \\ -2 & 1-x \end{vmatrix} = (1-x)^2 + 4 = x^2 - 2x + 1 + 4 = x^2 - 2x + 5$. The eigenvalues are the roots of this polynomial. Using the quadratic formula, the roots are $\frac{2 \pm \sqrt{(-2)^2 - 4*1*5}}{2} = \frac{2 \pm \sqrt{-16}}{2} = \frac{2 \pm 4i}{2} = 1 \pm 2i$.

(b) The eigenvalues of an echelon form matrix are simply the diagonal entries, so the eigenvalues of this matrix are $3+i$, 2, and $-i$.

(c) $\begin{vmatrix} 5-x & 0 & 0 \\ -2 & 3-x & 6 \\ 0 & 1 & -2-x \end{vmatrix}$

$= (5-x) \begin{vmatrix} 3-x & 6 \\ 1 & -2-x \end{vmatrix} = (5-x)\big((3-x)(-2-x) - 6\big) = (5-x)(-6 - x + x^2 - 6)$

$= (5-x)(x^2 - x - 12) = (5-x)(x-4)(x+3)$.

The eigenvalues are the roots of this polynomial, 5, 4, and -3.

Problem 3. Suppose that A is an $n \times n$ matrix, and λ is an eigenvalue for A. Show that λ^2 is an eigenvalue for AA. [Hint: Let $\vec{v}$ be an eigenvector for A with eigenvalue λ.]

Answer:

Let $\vec{v}$ be an eigenvector for A with eigenvalue λ. Then $A\vec{v} = \lambda v$. We then have $(AA)\vec{v} = A(A\vec{v}) = A(\lambda\vec{v}) = \lambda \cdot A\vec{v} = \lambda \cdot \lambda\vec{v} = (\lambda \cdot \lambda)\vec{v} = \lambda^2\vec{v}$. So λ^2 is an eigenvalue for the matrix AA (and $\vec{v}$ is an eigenvector for AA with eigenvalue λ^2.

Problem 4. Let $h\colon \mathbb{C}^2 \to \mathbb{C}^2$ be a homomorphism that has eigenvalues $1-i$ and $1+i$. Suppose that $\begin{pmatrix} 1 \\ -2 \end{pmatrix}$ is an eigenvector with eigenvalue $1-i$, and that $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ is an eigenvector with eigenvalue $1+i$. Find the matrix for h in the standard basis, $\langle \vec{e}_i, \vec{e}_2 \rangle$. [Hint: You need to find $h(\vec{e}_1)$ and $h(\vec{e}_2)$.]

Answer:

The matrix for h in the standard basis is $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ where $h\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} a \\ c \end{pmatrix}$ and $h\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} b \\ d \end{pmatrix}$. So we just need to compute those values.

$$\begin{aligned} h\begin{pmatrix} 1 \\ 0 \end{pmatrix} &= h\left((1/3) \cdot \begin{pmatrix} 1 \\ -2 \end{pmatrix} + (2/3) \cdot \begin{pmatrix} 1 \\ 1 \end{pmatrix}\right) \\ &= (1/3) \cdot h\begin{pmatrix} 1 \\ -2 \end{pmatrix} + (2/3) \cdot h\begin{pmatrix} 1 \\ 1 \end{pmatrix} \\ &= (1/3) \cdot (1-i) \cdot \begin{pmatrix} 1 \\ -2 \end{pmatrix} + (2/3) \cdot (1+i)\begin{pmatrix} 1 \\ 1 \end{pmatrix} \\ &= \begin{pmatrix} 1 + \frac{1}{3}i \\ \frac{4}{3}i \end{pmatrix} \end{aligned}$$

and

$$\begin{aligned}
h\begin{pmatrix}0\\1\end{pmatrix} &= h\left(-(1/3)\cdot\begin{pmatrix}1\\-2\end{pmatrix}+(1/3)\cdot\begin{pmatrix}1\\1\end{pmatrix}\right)\\
&= -(1/3)\cdot h\begin{pmatrix}1\\-2\end{pmatrix}+(1/3)\cdot h\begin{pmatrix}1\\1\end{pmatrix}\\
&= -(1/3)\cdot(1-i)\cdot\begin{pmatrix}1\\-2\end{pmatrix}+(1/3)\cdot(1+i)\begin{pmatrix}1\\1\end{pmatrix}\\
&= \begin{pmatrix}\frac{2}{3}i\\1-\frac{1}{3}i\end{pmatrix}
\end{aligned}$$

So the matrix in the standard basis is $\begin{pmatrix}1+\frac{1}{3}i & \frac{2}{3}i\\ \frac{4}{3}i & 1-\frac{1}{3}i\end{pmatrix}$.

Problem 5. Let $\mathscr{D}$ be the vector space of differentiable functions from $\mathbb{R}$ to $\mathbb{R}$. That is, $\mathscr{D}$ is the set $\{f\colon \mathbb{R}\to\mathbb{R}\,|\, f'(x) \text{ exists for all } x\}$, with the usual addition and scalar multiplication for real-valued functions. Let $\partial\colon \mathscr{D}\to\mathscr{D}$ be the derivative function, $\partial(f)=f'$. Show that every real number λ is an eigenvalue for ∂, and find an eigenvector for eigenvalue λ. [Hint: What is the derivative of e^{ax}? Once you remember that derivative, this question is trivial.]

Answer:

Let $\lambda\in\mathbb{R}$. For $\lambda=0$, the function $f(x)=1$ is an eigenvector for ∂ with eigenvalue 0, because $\partial(f)=0=0\cdot f$. For $\lambda\neq 0$, let $g(x)=e^{\lambda x}$. Then $g'(x)=\frac{d}{dx}e^{\lambda x}=\lambda e^{\lambda x}=\lambda g(x)$. That is, $\partial(g)=\lambda g$. So, $g(x)=e^{\lambda x}$ is an eigenvector for ∂ with eigenvalue λ. [Note that the last statement is true even for $\lambda=0$ since $e^{0x}=e^0=1$.]