Problem 1. The following systems of linear equations have unique solutions. Use Gauss's Method to put each system to echelon form, and then find the solution. As you apply row operations, show the result of each operation and show which operation you are applying. You can specify a row operation using the same notation as the textbook. You can combine several row operations of the form $\rho_j + k\rho_i$ into one step, as long as they use the same ρ_i.

(a)
$$\begin{array}{rrcr} 2x & -3y & = & -1 \\ x & +2y & = & 3 \end{array}$$

(b)
$$\begin{array}{rrrcr} & x_2 & +2x_3 & = & 3 \\ x_1 & -x_2 & -3x_3 & = & -2 \\ 2x_1 & +4x_2 & -x_3 & = & 0 \end{array}$$

(c)
$$\begin{array}{rrrcr} x & +y & +z & = & 1 \\ x & -y & -2z & = & 2 \\ 2x & +y & +z & = & 3 \\ x & -y & & = & 4 \end{array}$$

(a)

$$\begin{array}{rrcr} 2x & -3y & = & -1 \\ x & +2y & = & 3 \end{array} \xrightarrow{-\frac{1}{2}\rho_1 + \rho_2} \begin{array}{rrcr} 2x & -3y & = & -1 \\ & \frac{7}{2}y & = & \frac{7}{2} \end{array}$$

From the second equation, we get $y = 1$. The the first equation gives $2x = -1 + 3y = -1 + 3 = 2$, so $x = 1$. The unique solution is $x = 1, y = 1$. (Note that the solution can be checked by plugging these values into the original equations.)

(b)

$$\begin{array}{rrrcr} & x_2 & +2x_3 & = & 3 \\ x_1 & -x_2 & -3x_3 & = & -2 \\ 2x_1 & +4x_2 & -x_3 & = & 0 \end{array} \xrightarrow{\rho_1 \leftrightarrow \rho_2} \begin{array}{rrrcr} x_1 & -x_2 & -3x_3 & = & -2 \\ & x_2 & +2x_3 & = & 3 \\ 2x_1 & +4x_2 & -x_3 & = & 0 \end{array}$$

$$\xrightarrow{-2\rho_1 + \rho_3} \begin{array}{rrrcr} x_1 & -x_2 & -3x_3 & = & -2 \\ & x_2 & +2x_3 & = & 3 \\ & 6x_2 & 5x_3 & = & 4 \end{array}$$

$$\xrightarrow{-6\rho_1 + \rho_3} \begin{array}{rrrcr} x_1 & -x_2 & -3x_3 & = & -2 \\ & x_2 & +2x_3 & = & 3 \\ & & -7x_3 & = & -14 \end{array}$$

The last equation gives $x_3 = 2$. Then the second equation gives $x_2 = 3 - 3x_3 = 3 - 4 = -1$. Finally, the first equation gives $x_1 = -2 + x^2 + 3x_3 = -2 - 1 + 6 = 3$. So the unique solution is $x_1 = 3$, $x_2 = -1$, $x_3 = 2$.

(c)

$$\begin{array}{rrrcr} x & +y & +z & = & 1 \\ x & -y & -2z & = & 2 \\ 2x & +y & +z & = & 3 \\ x & -y & & = & 4 \end{array} \xrightarrow[-\rho_1+\rho_4]{\substack{-\rho 1 + \rho 2 \\ -2\rho_1+\rho_3}} \begin{array}{rrrcr} x & +y & +z & = & 1 \\ & -2y & -3z & = & 1 \\ & -y & -z & = & 1 \\ & -2y & -z & = & 3 \end{array}$$

$$\xrightarrow{\rho_2 \leftrightarrow \rho_3} \begin{array}{rrrcr} x & +y & +z & = & 1 \\ & -y & -z & = & 1 \\ & -2y & -3z & = & 1 \\ & -2y & -z & = & 3 \end{array}$$

$$\xrightarrow[-2\rho_2+\rho_4]{-2\rho_2+\rho_3} \begin{array}{rrrcr} x & +y & +z & = & 1 \\ & -y & -z & = & 1 \\ & & -z & = & -1 \\ & & z & = & 1 \end{array}$$

$$\xrightarrow{\rho_3+\rho_4} \begin{array}{rrrcr} x & +y & +z & = & 1 \\ & -y & -z & = & 1 \\ & & -z & = & -1 \\ & & 0 & = & 0 \end{array}$$

The final $0 = 0$ row is always true. The third row gives $z = 1$. Then the second row gives $y = z + 1 = 2$. Then the first row gives $x = 1 - y - z = 1 - 2 - 1 = -2$. So, the solution is $x = 2$, $y = -2$, $z = 1$,

Problem 2. For each of the linear systems in problem **1**, rewrite the system in the form of an augmented matrix. For this short problem, *you do not need to show any work, just write the answers*. You just have to write the augmented matrix form of the original system of equations.

(a) $$\left(\begin{array}{cc|c} 2 & -3 & -1 \\ 1 & 2 & 3 \end{array}\right)$$

(b) $$\left(\begin{array}{ccc|c} 0 & 1 & 2 & 3 \\ 1 & -1 & -3 & -2 \\ 2 & 4 & -1 & 0 \end{array}\right)$$

(c)
$$\left(\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 1 & -1 & -2 & 2 \\ 2 & 1 & 1 & 3 \\ 1 & -1 & 0 & 4 \end{array}\right)$$

Problem 3. The following systems are already in echelon form. Each each system has an infinite number of solutions. Express the set of solutions in vector form. The answers will have a form similar to $\{\vec{v}_1 + a\vec{v}_2 + b\vec{v}_3 \,|\, a, b \in \mathbb{R}\}$, where v_1, v_2 and v_3 are column vectors of constants.

(a)
$$\begin{array}{rrrcr} x & -3y & -z & = & -1 \\ & 2y & +3z & = & 5 \end{array}$$

(b)
$$\begin{array}{rrrrrcr} 2x_1 & -x_2 & +3x_3 & +x_4 & -2x_5 & = & 3 \\ & & -x_3 & +2x_4 & -x_5 & = & 1 \\ & & & x_4 & +4x_5 & = & -2 \end{array}$$

(a) z is a free variable in this system, so it can have any value. To find the solution set:

$$z = a$$

$$\begin{aligned} 2y &= 5 - 3z \\ y &= \frac{5}{2} - \frac{3}{2}z \\ &= \frac{5}{2} - \frac{3}{2}a \end{aligned}$$

$$\begin{aligned} x &= -1 + 3y + z \\ &= -1 + 3(\frac{5}{2} - \frac{3}{2}a) + a \\ &= -1 + \frac{15}{2} - \frac{9}{2}a + a \\ &= \frac{13}{2} - \frac{7}{2} \end{aligned}$$

The solution set is

$$\left\{ \begin{pmatrix} \frac{13}{2} - \frac{7}{2}a \\ \frac{5}{2} - \frac{3}{2}a \\ a \end{pmatrix} \,:\, a \in \mathbb{R} \right\} = \left\{ \begin{pmatrix} \frac{13}{2} \\ \frac{5}{2} \\ 0 \end{pmatrix} + a \begin{pmatrix} -\frac{7}{2} \\ -\frac{3}{2} \\ 1 \end{pmatrix} \,:\, a \in \mathbb{R} \right\}$$

(b) x_2 and x_5 are free variables in this system, so they can have any value

$$x_5 = b$$

$$\begin{aligned} x_4 &= -2 - 4x_5 \\ &= -2 - 4b \end{aligned}$$

$$\begin{aligned} -x_3 &= 1 - 2x_4 + x_5 \\ x_3 &= -1 + 2x_4 - x_5 \\ &= -1 + 2(-2 - 4b) - b \\ &= -5 - 9b \end{aligned}$$

$$x_2 = a$$

$$\begin{aligned} 2x_1 &= 3 + x_2 - 3x_3 - x_4 + 2x_5 \\ &= 3 + a - 3(-5 - 9b) - (-2 - 4b) + 2b \\ &= 3 + a + 15 + 27b + 2 + 4b + 2b \\ &= 20 + a + 33b \\ x_1 &= 10 + \frac{1}{2}a + \frac{33}{2}b \end{aligned}$$

The solution set is:

$$\left\{ \begin{pmatrix} 10 + \frac{1}{2}a + \frac{33}{2}b \\ a \\ -5 - 9b \\ -2 - 4b \\ b \end{pmatrix} \;:\; a, b \in \mathbb{R} \right\} = \left\{ \begin{pmatrix} 10 \\ 0 \\ -5 \\ -2 \\ 0 \end{pmatrix} + a \begin{pmatrix} \frac{1}{2} \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} + b \begin{pmatrix} \frac{33}{2} \\ 0 \\ -9 \\ -4 \\ 1 \end{pmatrix} \;:\; a, b \in \mathbb{R} \right\}$$

Problem 4. The following augmented matrix represents a system of linear equations:

$$\left(\begin{array}{ccc|c} 1 & 1 & 1 & 3 \\ 2 & 3 & -1 & 4 \\ 1 & 2 & -2 & b \end{array}\right)$$

For which values of the variable b, if any, does the system have exactly one solution? no solution? infinitely many solutions?

Apply Gauss's method to the matrix:

$$\left(\begin{array}{ccc|c} 1 & 1 & 1 & 3 \\ 2 & 3 & -1 & 4 \\ 1 & 2 & -2 & b \end{array}\right) \xrightarrow[-\rho 1 + \rho_3]{-2\rho_1 + \rho_2} \left(\begin{array}{ccc|c} 1 & 1 & 1 & 3 \\ 0 & 1 & -3 & -2 \\ 0 & 1 & -3 & b-3 \end{array}\right)$$

$$\xrightarrow{-\rho_2 + \rho_3} \left(\begin{array}{ccc|c} 1 & 1 & 1 & 3 \\ 0 & 1 & -3 & -2 \\ 0 & 0 & 0 & b-1 \end{array}\right)$$

If $b-1=0$, then the last row is $0=0$, and the system has infinite number of solutions because there is one free variable. If $b-1 \neq 0$, then the last row is of the form $0=k$ where $k \neq 0$, so there is no solution. Those are the only possibilities. So either $b=1$, and there is an infinite number of solutions, or $b \neq 5$ and there is no solution.

There can never be exactly one solution, there is no solution if b is any number besides 1, and there is an infinite number of solutions if b is 1.