Math 135, Fall 2019, Homework 11

This homework is due the last day of class, Monday, December 8.

1. (Textbook exercise 14.1.10.) Find a bijection from the set $\{0, 1\} \times \mathbb{N}$ to $\mathbb{Z}$. (Prove that the function that you find is bijective.)

2. (Textbook exercise 14.2.4.) Prove that the set of all irrational numbers is uncountable. (Hint: Consider a proof by contradiction using Theorems 14.4 and 14.6.)

3. (Textbook exercise 14.3.4.) Prove or disprove: If $A \subseteq B \subseteq C$ and A and C are countably infinite, then B is countable infinite. (Hint: See Theorem 14.8.)

4. (Textbook exercise 14.3.8.) Prove or disprove: The set $\{(a_1, a_2, a_3, \dots) : a_i \in \mathbb{N}\}$ of infinite sequences of natural numbers is countably infinite. (Hint: Consider just the sequences in which all of the numbers are zeros and ones.)

5. Textbook exercise 14.2.13 is "Prove or disprove: Let $A = \{X \subseteq \mathbb{N} : X \text{ is finite}\}$. Then $|A| = \aleph_0$." The solution in the back of the book proves that yes, A is countably infinite, but it uses a complicated procedure to put the elements of A into a list. Describe how to get a simpler listing by first ordering the subsets of $\mathbb{N}$ according the the sum of the elements of the set. So, the empty set comes first with sum zero, followed by the set $\{1\}$ with sum 1, followed by the set $\{2\}$ with sum 2, followed by the sets $\{3\}$ and $\{1, 2\}$ with sum 3. Pay attentition to how all of the sets whose elements sum up to a given number n will be ordered! List the first 16 sets in your list of sets.

6. Let A be an infinite set, and let $x \in A$. Show that there is a bijection from A to $A \setminus \{x\}$. (Hint: A must have a countably infinte subset, which can be listed as $\{a_1, a_2, a_3, \dots\}$.)

7. (Textbook exercise 14.2.15.) Theorem 14.5 implies that $\mathbb{N} \times \mathbb{N}$ is countably infinite. Construct an alternate proof of this fact by showing that the function $\varphi\colon \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ defined by $\varphi(m, n) = 2^{n-1}(2m - 1)$ is bijective. (The solution at the back of the book only gives a hint: Use the Fundamental Theorem of Arithmetic.)

8. (Textbook exercise 14.2.12.) Exercise 14.2.11 asks for a partition of $\mathbb{N}$ into eight countably infinite sets. That can be done simply by looking at the equivalence classes of the relation $\equiv \pmod 8$. Exercise 14.2.12 asks you to describe a partition of $\mathbb{N}$ into $\aleph_0$ contably infinite sets (that is, a countably infinite number of countably infinite sets), which is harder. But not so hard if you keep in mind that there is a bijection between $\mathbb{N} \times \mathbb{N}$ and $\mathbb{N}$.