Math 135, Fall 2019, Sample Solutions for Homework 11

1. Define the function $f\colon \{0,1\}\times\mathbb{N}\to\mathbb{Z}$ by $f(x,n)=\begin{cases} n-1 & \text{if } x=0\\ -n & \text{if } x=1\end{cases}$. To see that f is bijective, define $g\colon \mathbb{N}\to\{0,1\}\times\mathbb{N}$ by $g(n)=\begin{cases}(0,n+1) & \text{if } n\ge 0\\ (1,-n) & \text{if } n<0\end{cases}$. We show that g is an inverse function for f, which also proves that f is a bijection. We need to show that $g(f(a,n))=(a,n)$ for all $(a,n)\in\{0,1\}\times\mathbb{N}$, and that $f(g(n))=n$ for all $n\in\mathbb{N}$.

Let $(a,n)\in\{0,1\}\times\mathbb{N}$. We show $g(f(a,n))=(a,n)$. In the case $a=0$, then $g(f(0,n))=g(n-1)=(n-1)+1=n$ (using the fact that, since $n\in\mathbb{N}$, $n-1\ge 0$). In the case $a=1$, then $g(f(1,n))=g(-n)=(1,-(-n))=(1,n)$.

Let $n\in\mathbb{Z}$. We show $f(g(n))=n$. In the case $n\ge 0$, $n+1\in\mathbb{N}$ and $f(g(n))=f(0,n+1)=(n+1)-1=n$. In the case $n<0$, then $-n>0$ and $f(g(n))=f(1,-n)=-(-n)=n$.

2. Let I be the set of all irrational numbers, so that $I=\mathbb{R}\smallsetminus\mathbb{Q}$. Suppose, for the sake of contradiction, that I is countable. By Theorem 14.4, $\mathbb{Q}$ is countable. Since $\mathbb{R}=\mathbb{Q}\cup I$ and both $\mathbb{Q}$ and I are countable, then by Theorem 14.6, $\mathbb{R}$ is countable. But in fact, we know by Theorem 14.2 that $\mathbb{R}$ is uncountable, not countable. This contradiction proves that I must be uncountable.

3. Suppose that A, B, and C are sets, that $A\subseteq B\subseteq C$, and that both A and B are countably infinite. Since $A\subseteq B$ and A is infinite, we know that B must also be infinite (since a subset of a finite set would be finite). Since C is countably infinite and B is an infinite subset of C, then by Theorem 14.8, B is countably infinite.

4. If the set $\{(a_1,a_2,a_3,\dots):a_i\in\mathbb{Z}\}$ were countably infinite then, by Theorem 14.8, its subset $X=\{(a_1,a_2,a_3,\dots):a_i\in\{0,1\}\}$ would also be countably infinite. However, we show that X is uncountable by finding a bijection of that set with $\mathscr{P}(\mathbb{N})$. In fact, define the function $f\colon X\to\mathscr{P}(\mathbb{N})$ by $f((a_1,a_2,a_3,\dots))=\{n\in\mathbb{N}:a_n=1\}$. This function has inverse function $g\colon\mathscr{P}(\mathbb{N})\to X$ defined by $g(A)=(b_1,b_2,b_3,\dots)$, where $b_n=1$ if and only if $n\in A$. (It is clear from the definitions that $f\circ g$ and $g\circ f$ are identity functions.)

5. As suggested by the statement of the problem, we can put the finite subsets of $\mathbb{N}$ into an infinite list that is ordered as follows: Order the list of subsets first by the sum of the elements in the subset. For subsets with the same sum, order them "lexicographically". That is, to compare two sets with the same sum, list the elements of the sets in order from smallest to largest, the order them by comparing their first elements, or if their first elements are equal the by comparing their second elements, or if their first and second elements are equal, then by comparing their third elements, and so on. Using this ordering, the first 16 sets in the list are:

$\{\}$	$\{1\}$	$\{2\}$	$\{1,2\}$
$\{3\}$	$\{1,3\}$	$\{4\}$	$\{1,4\}$
$\{2,3\}$	$\{5\}$	$\{1,2,3\}$	$\{1,5\}$
$\{2,4\}$	$\{6\}$	$\{1,2,4\}$	$\{1,6\}$

6. Let A be an infinite set, and let $x \in A$. We can find an infinite subset $B = \{a_1, a_2, a_3, \dots\}$, and we can assume that x is **not** in that subset. (If x is in B, we can remove it.) Then we can define a bijection $f : A \to A \smallsetminus \{x\}$ by letting $f(x) = a_1$, $f(a_n) = a_{n+1}$ for $n \in \mathbb{N}$, and $f(y) = y$ for any other elements of A such that $y \neq x$ and $y \notin \{a_1, a_2, a_3, \dots\}$.

7. We note that any natural number $x \in \mathbb{N}$ can be written uniquely as $x = 2^{n-1}m$ where $n \in \mathbb{N}$ and m is an odd natural number. This is because $1 = 2^{1-0} \cdot 1$, and for any natural number $x > 1$, x can be factored uniquely into a product of prime numbers, $x = p_1 p_2 \cdots p_k$. We can separate out the factors that are equal to 2, giving 2^{n-1} for some $n \in \mathbb{N}$. (Using $n-1$ as the exponent allows the case 2^0 which occurs when x does not have any prime factors equal to 2.) The remaining prime factors are all odd numbers, so their product is also an odd number. That is, we have written $x = 2^{n-1}k$, where k is odd. Since any odd natural number k can be written $k = 2m-1$ for some $m \in \mathbb{N}$, we see that every natural number can be written in the form $2^{n-1}(2m-1)$ for some $n, m \in \mathbb{N}$. Furthermore, this representation is unique. That is, if $2^{n-1}(2m-1) = 2^{k-1}(2\ell-1)$, then $n = k$ and $m = l$. The uniqueness follows from the fundamental theorem of arithmetic, since the power of 2 in the representation of a number x is just the number of times 2 occurs as a prime factor in x, and the odd number in the representation is the product of all the odd primes that occur in the prime factorization.

Now, suppose $\varphi\colon \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ is defined by $\varphi(m, n) = 2^{n-1}(2m-1)$. Since any number $x \in \mathbb{N}$ can be written in the form $2^{n-1}(2m-1)$ for some $n, m \in \mathbb{N}$, it follows that φ is surjective. Since the representation is unique, it follows that φ is injective. So, we see that φ is a bijection.

8. For each natural number k, let $A_k = \{(2k-1), 2(2k-1), 2^2(2k-1), 2^3(2k-1), 2^4(2k-1), \dots\}$. Since every $x \in \mathbb{N}$ can be represented uniquely in the form $x = 2^{n-1}(2m-1)$, as seen in the solution to the previous problem, the sets A_k form a partition of $\mathbb{N}$ (that is every $x \in \mathbb{N}$ is in one and only one of the sets). Furthermore, each set A_k is countably infinite, and there are a countably infinite number of them.