1. Define the word *tautology* (as it applies to propositional logic).

Answer:

A tautology is a proposition that is true for all possible values of the propositional variables that it contains.

2. Use a **truth table** to verify the logical equivalence: $p \vee ((\neg p) \wedge q) \equiv (\neg q) \rightarrow p$. (What about the truth table shows that these propositions are logically equivalent?)

Answer:

p	q	$\neg p$	$(\neg p) \wedge q$	$p \vee ((\neg p) \wedge q)$	$\neg q$	$(\neg q) \rightarrow p$
T	T	F	F	T	F	T
T	F	F	F	T	T	T
F	T	T	T	T	F	T
F	F	T	F	F	T	F

Since the last two columns are identical, $p \vee ((\neg p) \wedge q) \equiv (\neg q) \rightarrow p$

3. Simplify the following, so that in the end the $\neg$ operator is applied only to individual predicates. (Show the steps in the simplification.)

$$\neg\big[\,\forall x\,\big(P(x) \rightarrow (\exists y\, R(x,y))\big)\,\big]$$

Answer:

$$\begin{aligned}
\neg\big[\,\forall x\,\big(P(x) \rightarrow (\exists y\, R(x,y))\big)\,\big] &\equiv \exists x\,\neg\big(P(x) \rightarrow (\exists y\, R(x,y))\big) \\
&\equiv \exists x\,\big(P(x) \wedge (\neg(\exists y\, R(x,y)))\big) \\
&\equiv \exists x\,\big(P(x) \wedge (\forall y\, \neg R(x,y))\big)
\end{aligned}$$

4. Consider the statement, "If Spring is here, then I am happy."

a) State the *contrapositive* of this statement in natural English.

b) State the *negation* of this statement in natural English.

Answer:

a) If I am not happy, then Spring is not here.

b) Spring is here, but I am not happy.

5. Give a *formal proof* that the following argument is valid. (State a reason for each step in the proof.)

$$\begin{array}{l} p \to (q \vee r) \\ (t \wedge p) \to (\neg q) \\ s \to t \\ s \\ p \\ \hline \therefore r \end{array}$$

Answer:

(a)

1.	$s \to t$	(premise)
2.	s	(premise)
3.	t	(from 1 and 2, my Modus Ponens)
4.	p	(premise)
5.	$t \wedge p$	(from 3 and 4, definition of $\wedge$)
6.	$(t \wedge p) \to (\neg q)$	(premise)
7.	$\neg q$	(from 5 and 6, by Modus Ponens)
8.	$p \to (q \vee r)$	(premise)
9.	$q \vee r$	(from 4 and 8, by Modus Ponens)
10.	r	(from 7 and 9, by Elimination)

6. Consider the following propositions, where the domain of discourse in all cases is the set of people:

$R(x)$ stands for "x is rich"
$H(x)$ stands for "x is happy"
$L(u, v)$ stands for "u likes v"

a) Translate the sentence "Everyone is rich and happy" into predicate logic.

b) Translate the sentence "All rich people are happy" into logic.

c) Translate the sentence "There is an unhappy rich person" into logic.

d) Express the proposition $\forall x \big(R(x) \to \forall y \, L(y, x)\big)$ as a sentence in natural English.

Answer:

a) $\forall x \big(R(x) \wedge H(X)\big)$

b) $\forall x \big(R(x) \to H(X)\big)$

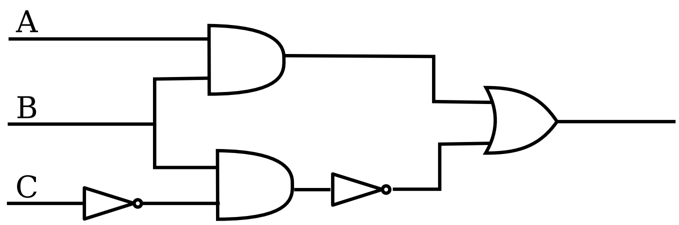
c) $\exists x \big(R(x) \wedge (\neg H(X))\big)$

d) If you are rich, everyone likes you. (Another possibility: Everyone who is rich is liked by everyone.)

7. Draw the logic circuit that computes the following boolean expression:

$$(A \wedge B) \vee (\neg(B \wedge \neg C))$$

Answer:



8. Recall that $\mathbb{N} = \{0, 1, 2, 3, 4, \ldots\}$. Let $\mathbb{E} = \{0, 2, 4, 6, \ldots\}$, the set of natural numbers that are even.

a) Write out the set $\mathbb{E} \cap \{0, 1, 4, 9, 16, 25, 36, 49\}$

b) Identify the set $\mathbb{N} \smallsetminus \mathbb{E}$

c) Write out the set $\{\, x \in \mathbb{E} \,|\, x < 10 \,\}$

Answer:

a) $\{0, 4, 16, 36\}$

b) It is the set of all odd natural numbers, $\{1, 3, 5, 9, 11, \ldots\}$

c) $\{0, 2, 4, 6, 8\}$

9. Suppose that 16-bit binary numbers are used to represent subsets of $\{15, 14, \ldots, 1, 0\}$.

a) What set is represented by 1010 0100 1100 0001 ?

b) What 16-bit number represents the set $\{12, 6, 5, 3, 2\}$?

c) The left shift operator does not implement a set operation. But suppose that `m` is a 16-bit binary number representing a set, A. What numbers would be in the set represented by `m << 1` compared to the numbers in the set A? Why? (Consider the sets in parts **a** and **b** as examples!)

Answer:

a) $\{15, 13, 10, 7, 6, 0\}$

b) 0001 0000 0110 1100

c) `m << 1` represents the set whose elements are the elements of A incremented by 1, except that if 15 is one of the elements of A, 16 is not in the set represented by `m << 1`. That is, the set is $\{n + 1 \mid n \in A \wedge n \neq 15\}$. This is because `m << 1` shifts each 1 in `m` one position to the left. In that position, the bit rerpresents a number that is one more than the number represented by the 1 in its previous position. However, if there is a 1 in the leftmost position, representing the number 15, it is lost when it is shifted one position to the left, so it does not contribute any element to `m << 1`.

10. **a)** Define *subset*. (That is, what does it mean to say $A \subseteq B$.)
b) Define *power set* of a set.

Answer:

a) Let A and B be sets. We say that A is a subset of B if every element of A is also an element of B. (More symbolically, $A \subseteq B$ if and only if $\forall x\,(x \in A \rightarrow x \in B)$.)

b) The power set of a set A is the set whose elements are all of the subsets of A. (More symbolically, the powers set $\mathscr{P}(A)$ is defined as $\mathscr{P}(A) = \{X \mid X \subseteq A\}$.)

11. Prove the following statement: For any integers n and m, if n and m are odd, then $n + m$ is even.

Answer:

Let n and m be arbitrary integers. Suppose that n and m are odd. Since n is odd, then by definition, there is an integer k such that $n = 2k + 1$. Since m is odd, then by definition, there is an integer j such that $m = 2j + 1$. We then have $n + m = (2k + 1) + (2j + 1) = 2k + 2j + 2 = 2(k + j + 1)$. Since $k + j + 1$ is an integer, this means by definition of even that $n + m$ is even.

12. Use a proof by induction to show that for any integer $k \geq 0$, $\displaystyle\sum_{i=0}^{k} 2^i = 2^{k+1} - 1$

Answer:

Base Case. For $k = 0$, the statement is $\displaystyle\sum_{i=0}^{0} 2^i = 2^{0+1} - 1$. since $\displaystyle\sum_{i=0}^{0} 2^i = 2^0 = 1$, and $2^{0+1} - 1 = 2 - 1 = 1$, the statement is true in the base case.

Inductive Case. Let $k \geq 0$ and assume that the statement is true for k. We must show that the statement is true for $k + 1$. That is, assume $\displaystyle\sum_{i=0}^{k} 2^i = 2^{k+1} - 1$, and prove $\displaystyle\sum_{i=0}^{k+1} 2^i = 2^{(k+1)+1} - 1 = 2^{k+2} - 1$. But

$$\begin{aligned}
\sum_{i=0}^{k+1} 2^i &= \left(\sum_{i=0}^{k} 2^i\right) + 2^{k+1} \\
&= \left(2^{k+1} - 1\right) + 2^{k+1} \\
&= 2^{k+1} + 2^{k+1} - 1 \\
&= 2 \cdot 2^{k+1} - 1 \\
&= 2^{k+2} - 1
\end{aligned}$$

which completes the inductive case and the proof.