1. (*3 points*) Write out each of the following sums in full, without using summation notation. You are **not** being asked to compute the value of the sums, just to expand each sum into a normal sum of several terms!

a) $\sum_{n=1}^{5} n \cdot 5^n$ **b)** $\sum_{i=1}^{7} (2i-1)$ **c)** $\sum_{k=3}^{6} \frac{k}{k^2+1}$

Answer:

a) $\sum_{n=1}^{5} n \cdot 5^n = 1 \cdot 5^1 + 2 \cdot 5^2 + 3 \cdot 5^3 + 4 \cdot 5^4 + 5 \cdot 5^5$

b) $\sum_{i=1}^{7} (2i-1) = (2\cdot 1-1)+(2\cdot 2-1)+(2\cdot 3-1)+(2\cdot 4-1)+(2\cdot 5-1)+(2\cdot 6-1)+(2\cdot 7-1)$

c) $\sum_{k=3}^{6} \frac{k}{k^2+1} = \frac{3}{3^2+1} + \frac{4}{4^2+1} + \frac{5}{5^2+1} + \frac{6}{6^2+1}$

2. (*3 points*) Use a proof by induction to show that for any integer $n \geq 1$, $\sum_{i=1}^{n} (2i-1) = n^2$

Answer:

Base Case: Let $n = 1$. Then $\sum_{i=1}^{n} (2i-1) = \sum_{i=1}^{1} (2i-1) = (2 \cdot 1 - 1) = 1$, and $n^2 = 1^2 = 1$. So the statement is true for the base case.

Inductive Case: Let $k \geq 1$, and assume that the statement is true for $n = k$. That is, assume that $\sum_{i=1}^{k} (2i-1) = k^2$. We must show that the statement holds for $n = k+1$, that is that $\sum_{i=1}^{k+1} (2i-1) = (k+1)^2$. But

$$\begin{aligned}
\sum_{i=1}^{k+1} (2i-1) &= \left(\sum_{i=1}^{k} (2i-1) \right) + (2 \cdot (k+1) - 1) \\
&= (k^2) + (2 \cdot (k+1) - 1) \\
&= k^2 + 2k + 2 - 1 \\
&= k^2 + 2k + 1 \\
&= (k+1)^2
\end{aligned}$$

3. (*3 points*) Use a proof by induction to show that for any integer $n \geq 1$, $n^3 + 2n$ is a multiple of 3. (That is, there is some integer j such that $n^3 + 2n = 3j$.)

Answer:

Base Case: Let $n = 1$. Then $n^3 + 2n = 1^3 + 2 \cdot 1 = 3 = 3 \cdot 1$. So $n^3 + 2n$ is a multiple of 3 in this case.

Inductive Case: Let $k \geq 1$, and assume that $n^3 + 2n$ is a multiple of 3 for $n = k$. That is, there is an integer j such that $k^3 + 2k = 3j$. We must show that $n^3 + 2n$ is a multiple of 3 for $n = k + 1$. But for $n = k + 1$, we have

$$\begin{aligned}
(k+1)^3 + 2(k+1) &= (k^3 + 3k^2 + 3k + 1) + (2k + 2) \\
&= (k^3 + 2k) + (3k^2 + 3k + 1 + 2) \\
&= (k^3 + 2k) + 3k^2 + 3k + 3 \\
&= 3j + 3k^2 + 3k + 3 \\
&= 3(j + k^2 + k + 1)
\end{aligned}$$

Since $j + k^2 + k + 1$ is an integer, this shows that $(k+1)^3 + 2(k+1)$ is a multiple of 3.

4. (*3 points*) Use a proof by induction to show that the following method correctly finds the sum of array elements `A[0]+ A[1]+ ··· + A[N-1]` for all $N \geq 1$.

```
int recursive_sum( int[] A, int N ) {
    if ( N == 1 )
        return A[0];
    else {
        return A[N-1] + recursive_sum( A, N-1 );
    }
}
```

Answer:

Base Case: Let $N = 1$. In this case, the test in the if statement is true, so the function returns `A[0]`. Since we are only taking the sum of one element, this is the correct answer.

Inductive Case: Suppose $k \geq 1$, and assume that the function is correct for $N = k$. That is, `recursive_sum(A,k)` correctly computes the sum `A[0]+A[1]+···+A[k-1]`. We want to show that `recursive_sum(A,k+1)` correctly computes the sum `A[0]+A[1]+···+A[k]`.

Since $k+1 > 1$, when `recursive_sum(A,k+1)` is called, the test in the if statement is false, and the return value of the function is computed as `A[(k+1)-1] + recursive_sum(A,(k+1)-1)`, which equals `A[k] + recursive_sum(A,k)`. By the inductive hypothesis, `recursive_sum(A,k)` correctly returns the value of `A[0]+A[1]+···+A[k-1]`. Then `A[k]` is added to that return value, giving the correct answer, `A[0]+A[1]+···+A[k-1]+A[k]`.

5. (*4 points*) Let $A = \{1,2,3,4,5,6,7,8,9\}$; let $B = \{2,4,6,8,10,12,14,16,18\}$; and let $C = \{n \in \mathbb{Z} \mid -5 \le n \le 5\}$. Find the following sets. (For this exercise, you do **not** need to justify your answers.)

a) $A \cup B$ **b)** $A \cap B$ **c)** $A \smallsetminus B$ **d)** $B \smallsetminus A$

e) $A \cap C$ **f)** $\mathbb{N} \cup C$ **g)** $\mathbb{N} \smallsetminus C$ **h)** $\mathbb{Z} \smallsetminus A$

(Recall that $\mathbb{N}$ is the set of natural numbers and $\mathbb{Z}$ is the set of integers.)

Answer:

a) $A \cup B = \{1,2,3,4,5,6,7,8,9,10,12,14,16,18\}$
b) $A \cap B = \{2,4,6,8\}$
c) $A \smallsetminus B = \{1,3,5,7,9\}$
d) $B \smallsetminus A = \{10,12,14,16,18\}$
e) $A \cap C = \{1,2,3,4,5\}$
f) $\mathbb{N} \cup C = \{n \in \mathbb{Z} \mid n \ge -5\} = \{-5,-4,-3,-2,-1,0,1,2,3,\dots\}$
g) $\mathbb{N} \smallsetminus C = \{n \in \mathbb{Z} \mid n > 5\} = \{6,7,8,9,10,11,\dots\}$
h) $\mathbb{Z} \smallsetminus A = \{n \in \mathbb{Z} \mid n \le 0 \text{ or } n \ge 10\}$

6. (*2 points*) Let A be the set $A = \{\, \varnothing, a, \{a\} \,\}$. Write out the power set, $\mathscr{P}(A)$. (It has 8 elements. You do not have to justify your answer.)

Answer:

$\mathscr{P}(A) = \{\, \varnothing, \{\varnothing\}, \{a\}, \{\{a\}\}, \{a, \{a\}\}, \{\varnothing, \{a\}\}, \{\varnothing, a\}, \{\varnothing, a, \{a\}\} \,\}$

7. (*3 points*) Let A be any set. What can you say about $A \cup A$? about $A \cap A$? about $A \smallsetminus A$? Justify your answer, either informally or by using the definitions of $\cup$, $\cap$, and $\smallsetminus$.

Answer:

$A \cup A = A$, because $A \cup A = \{x \mid x \in A \vee x \in A\} = \{x \mid x \in A\} = A$. This uses the idempotent law of logic for $\vee$. Informally, adding all the elements of A to all of the elements of A just gives all of the elements of A.

$A \cap A = A$, because $A \cap A = \{x \mid x \in A \wedge x \in A\} = \{x \mid x \in A\} = A$. This uses the idempotent law of logic for $\wedge$. Informally, taking all the elements that are in both A and A gives all of the elements of A.

$A \smallsetminus A = \varnothing$, because $A \smallsetminus A = \{x \mid x \in A \wedge \neg(x \in A)\} = \varnothing$, because $x \in A \wedge \neg(x \in A)$ is of the logical form $p \wedge \neg p$, which is always false. Informally, removing from A all of itself leaves nothing behind.

8. (*3 points*) Prove: If A, B, and C are any sets and $A \subseteq B$ and $A \subseteq C$, then $A \subseteq B \cap C$.

Answer:

Let A, B, and C be arbitrary sets, and assume that $A \subseteq B$ and $A \subseteq C$, We want to show that $A \subseteq B \cap C$. Saying $A \subseteq B \cap C$ means by definition that for any x, if $x \in A$, then $x \in B \cap C$.

Let x be an arbitrary element of A. We need to show that $x \in B \cap C$. Since $A \subseteq B$ and $x \in A$, we know that $x \in B$. Since $A \subseteq C$ and $x \in A$, we know that $x \in C$. Since $x \in B$ and $x \in C$, it follows by definition of intersection that $x \in B \cap C$.