1. (*6 points*) Translate each of the following arguments, expressed in English, into formal logic, and determine whether the argument is valid. If the argument is valid, give a formla proof of validity. If it is not valid, show that it is not valid.

a) In order to get a B.S. degree, Joe must take a math class or a computer science class. If Joe doesn't understand algebra, Joe can't take a math class. Joe has a B.S. degree, but Joe doesn't understand algebra. So Joe must have taken a computer science class.

b) If Bill stays up late partying, he is tired the next day. If Bill is tired and there is a test, he doesnt do well on the test. If Bill does well on a test, he celebrates. There was a test today, and Bill is not celebrating, so he must have stayed up late partying last night.

Answer:

a) Let D be "Joe gets a BS degree"; M, "Joe takes a math class"; C, "Joe takes a computer science class"; A, "Joe understands algebra". Then the argument translates as follows. The argument is valid, and a proof is given.

$$\begin{array}{l} D \to (M \vee C) \\ (\neg A) \to (\neg M) \\ D \\ \neg A \\ \hline \therefore\ C \end{array}$$

1 .	D	(premise)
2 .	$D \to (M \vee C)$	(premise)
3 .	$M \vee C$	(from 2,3 by Modus Ponens)
4 .	$\neg A$	(premise)
5 .	$(\neg A) \to (\neg M)$	(premise)
6 .	$\neg M$	(from 4,5 by Modus Ponens)
7 .	C	(from 3,6 by Elimination)

b) Let P be "Bill stayed up late partying"; R, "Bill is tired"; T, "There is a test"; W, "Bill does well on the test"; C, "Bill celebrates". Then the argument translates as follows. It is not valid, as shown by the assignment of truth values to the variables which makes all of the premises of the argument true and the conclusion false.

$$\begin{array}{l} P \to R \\ (R \wedge T) \to (\neg W) \\ W \to C \\ T \\ \neg C \\ \hline \therefore\ P \end{array}$$

P	false (Conclusion)
R	true
T	true (Premise)
W	false
C	false
$P \to R$	true (Premise)
$R \wedge T$	true
$\neg W$	true
$(R \wedge T) \to (\neg W)$	true (Premise)
$W \to C$	true
$\neg C$	true (Premise)

2. (*3 points*) Prove that for any integer n, the number $n^2 + n$ is even. [Hint: Consider a proof by cases, looking at the case where n is even and the case where n is odd.]

Answer:

Let n be an arbitrary integer. We know that n must be either even or odd.

Consider the case where n is even. In that case, by definition, there is an integer k such that $n = 2k$. Then $n^2 + n = (2k)^2 + (2k) = 4k^2 + 2k = 2(2k^2 + 1)$. Since $2k^2 + 1$ is an integer, this shows that $n^2 + n$ is even.

Consider the case where n is odd. In that case, by definition, there is an integer k such that $n = 2k+1$. Then $n^2+n = (2k+1)^2+(2k+1) = (4k^2+4k+1)+(2k+1) = 4k^2+6k+2 = 2(2k^2 + 3k + 1)$. Since $2k^2 + 3k + 1$ is an integer, this shows that $n^2 + n$ is even.

We have shown that for all cases, $n^2 + n$ is even.

3. (*3 points*) Suppose that x, y, and z are integers such that $x + y + z$ is greater than 30. Prove that at least one of x or y or z is greater than 10. [Hint: Consider a proof by contradiction.]

Answer:

Let x, y, and z be arbitrary integers, and assume that $x + y + z > 30$. Suppose, for the sake of contradiction, that it is not the case that at least one of x or y or z is greater than 10. (That is, suppose $\neg(x > 10 \vee y > 10 \vee z > 10)$.) This is equivalent, by DeMorgan's law, to saying that $x \leq 10$ and $y \leq 10$ and $z \leq 10$. But then $x+y+z \leq 10+10+10 = 30$. This contradicts the fact that $x + y + z > 30$, and this contradiction proves the theorem.

4. (*5 points*) Prove:

a) The product xy of any two rational numbers x and y is rational.

b) For any real number x, if x^2 is an irrational number, then x is also irrational. [Hint: Consider a proof by contrapositive.]

Answer:

a) Let x and y be arbitrary rational numbers. We want to show xy is rational. Since x is rational, there are integers a and b, where $b \neq 0$, such that $x = \frac{a}{b}$. Since y is rational, there are integers c and d, where $d \neq 0$, such that $y = \frac{c}{d}$. But then $xy = \left(\frac{a}{b}\right)\left(\frac{c}{d}\right) = \frac{ac}{bd}$. Since ac and bd are integers and $bd \neq 0$, this shows that xy is rational.

b) We prove the contrapositive: If x is not irrational, then x^2 is not irrational. That is, if x is rational, then x^2 is rational. Let x be an arbitrary rational number. By part **a**, xx is rational. That is x^2 is rational.

5. (*2 points*) Disprove: For any real number x, if x is an irrational number, then x^2 is also irrational.

Answer:

Let $x = \sqrt{2}$. We know that x is irrational. But $x^2 = (\sqrt{2})^2 = 2$, and 2 is a rational number. That is, x is irrational, but x^2 is not irrational. This counterexample proves that "For any real number x, if x is an irrational number, then x^2 is also irrational" is false.

6. (*3 points*) Prove: For any real number x, either x is irrational or $\pi - x$ is irrational.

Answer:

Proof by contradiction. Let x be an arbitrary real number. Suppose that it is **not** the case that x is irrational or $\pi - x$ is irrational. This is equivalent by DeMorgan's law to saying that x is not irrational and $\pi - x$ is not irrational. That is x is rational and $\pi - x$ is rational. Since the sum of two rational numbers is rational, it must then be the case that $x + (\pi - x)$ is rational. But $x + (\pi - x) = \pi$, which we know is not rational. This contradiction proves the theorem.

7. (*3 points*) Prove: Let a, b, and c be integers, and assume that $a|b$ and that $b|c$. Then $a|c$.

Answer:

Since $a|b$, there is an integer m such that $b = ma$. Since $b|c$, there is an integer k such that $c = kb$. But then $c = kb = k(ma) = (km)a$. Since km is an integer, this proves that $a|c$.