1. (*4* points) Draw truth tables to prove the following logical equivalences (and justify your answer by saying what it is about the table that proves equivalence):

a) $p \vee (q \vee p) \equiv p \vee q$

b) $(p \to r) \wedge (q \to r) \equiv (p \vee q) \to r$

Answer:

(a)

p	q	$q \vee p$	$p \vee (q \vee p)$	$p \vee q$
T	T	T	T	T
T	F	T	T	T
F	T	T	T	T
F	F	F	F	F

Since the last two columns are identical, $p \vee (q \vee p) \equiv p \vee q$

(b)

p	q	r	$p \to r$	$q \to r$	$(p \to r) \wedge (q \to r)$	$p \vee q$	$(p \vee q) \to r$
T	T	T	T	T	T	T	T
T	T	F	F	F	F	T	F
T	F	T	T	T	T	T	T
T	F	F	F	T	F	T	F
F	T	T	T	T	T	T	T
F	T	F	T	F	F	T	F
F	F	T	T	T	T	F	T
F	F	F	T	T	T	F	T

Since the last 6th and 8th columns are identical, $(p \to r) \wedge (q \to r) \equiv (p \vee q) \to r$

2. (*4* points) Now, prove the same logical equivalences using Boolean Algebra. That is, find a chain of logical equivalences leading from the left side of the equivalence to the right side, where a single definition or rule of Boolean algebra is applied in each step. For each step, state what definition or rule you are applying. (One rule that you will need is the fact that $a \to b \equiv (\neg a) \vee b$.)

a) $p \vee (q \vee p) \equiv p \vee q$

b) $(p \to r) \wedge (q \to r) \equiv (p \vee q) \to r$

Answer:

(a)
$$\begin{aligned} p \vee (q \vee p) &\equiv p \vee (p \vee q) && \text{(Commutative Law)} \\ &\equiv (p \vee p) \vee q && \text{(Associative Law)} \\ &\equiv p \vee q && \text{(Idempotent Law)} \end{aligned}$$

(b)
$$\begin{aligned} (p \to r) \wedge (q \to r) &\equiv ((\neg p) \vee r) \wedge (q \to r) && \text{(Definition of } \to) \\ &\equiv ((\neg p) \vee r) \wedge ((\neg q) \vee r) && \text{(Definition of } \to) \\ &\equiv ((\neg p) \wedge (\neg q)) \vee r && \text{(Distributive Law)} \\ &\equiv (\neg(p \vee q)) \vee r && \text{(DeMorgan's Law)} \\ &\equiv (p \vee q) \to r && \text{(Definition of } \to) \end{aligned}$$

3. (*2* points) Convert each of the following English statements into propositional logic. You should introduce symbols (such as p, q, d, f, etc.) to stand for the simple propositions that occur in the statements. State clearly what each symbol stands for. Try to express as much of the meaning of the sentence as possible.

a) *Jack is smart but not lucky.*

b) *If I have a choice, then I don't eat broccoli.*

Answer:

(a) Let s be "Jack is smart" and let k be "Jack is lucky." Then the sentence can be translated into propositional logic as: $s \wedge (\neg k)$

(b) Let c be "I have a choice" and let b be "I eat broccoli." Then the sentence can be translated into logic as: $c \rightarrow (\neg b)$

4. (*3* points) Consider the English statement, "*Achilles is brave and famous or long-lived.*" This statement is ambiguous. Give two translations into propositional logic. Then, for each translation into logic, give an unambiguous English statement with the same meaning.

Answer:

Let b be "Achilles is brave," let f be "Achilles is famous," and let v be "Achilles is long-lived."

The sentence could mean $(b \wedge f) \vee v$, which can be expressed unambiguously in English as "Either Achilles is brave and famous, or he is long-lived." (This is, of course, the correct translation, given Achilles' choice in the *Iliad*.)

Or the sentence could mean $b \wedge (f \vee v)$, which can be expressed unambiguously in English as "Achilles is brave, and he is either famous or long-lived." Applying DeMorgan's law, it could also be expressed as "Either Achilles is brave and famous, or he is brave and long-lived."

5. (*3* points) Express the negation of each of the following sentences in natural, unambiguous English.

a) *The answer is greater than ten and is less than or equal to 20.*

b) *Jack is smart but not lucky.*

c) *If Cassandra tells the truth, she will be believed.*

Answer:

(a) The answer is less that or equal to ten or is greater than 20. (The negation of $(a > 10) \wedge (a \leq 20)$ is $\neg(a > 10) \vee \neg(a \leq 20)$, by DeMorgan's law, which can be simplified to $(a \leq 10) \vee (a > 20)$.)

(b) Either Jack is not smart, or he is lucky. (The negation of $S \wedge \neg L$ is $(\neg S) \vee \neg(\neg L)$, which is equivalent to $(\neg S) \vee L$.)

(c) Cassandra tells the truth, but she is not believed. (The negation of $t \rightarrow b$ is $t \wedge \neg b$.)

6. (*4* points)

a) Give **(1)** the *converse*, **(2)** the *contrapositive*, and **(3)** the *negation* of the proposition:

$$p \rightarrow (\neg q)$$

b) Consider the statement, "*If Casey strikes out, then there is no joy in Mudville.*" Express in natural English **(1)** the *converse*, **(2)** the *contrapositive*, and **(3)** the *negation* of this statement.

Answer:

(a) Converse: $(\neg q) \rightarrow p$
Contrapositive: $(\neg(\neg q)) \rightarrow (\neg p)$, which simplifies to $q \rightarrow \neg p$
Negation: $p \wedge (\neg(\neg q))$, which simplifies to $p \wedge q$

(b) Converse: If there is no joy in Mudville, then Casey strikes out.
Contrapositive: If there is joy in Mudville, then Casey does not strike out.
Negation: Casey strikes out, but there is joy in Mudville.

(Logic doesn't deal well with time. In fact, Casey's at-bat comes before Mudville's reaction, so the real meanings could probably be better expressed as follows. Original: If Casey strikes out, then there will be no joy in Mudville; Converse: If there is no joy in Mudville, then Casey struck out; Contrapositive: If there is joy in Mudville, then Casey did not strike out; Negation: Casey struck out, but nevertheless there is joy in Mudville.)

7. (*5* points) Consider an ordinary poker deck of 52 playing cards. Using the definitions of the logical operators, determine the number of cards in the deck for which each of the following statements is logically true. Don't forget to justify each answer! The last two parts are tricky; you need to remember the logical meaning of $p \to q$.

a) "This card is both an Ace and a Spade"?
b) "This card is either an Ace or a Spade"?
c) "This card is an Ace but is not a Spade"?
d) "If this card is a Ace, then it is a Spade"?
e) "If this card is an Ace, then it is a King"?

Answer:

(a) 1 card. There is only one card that is both an ace and a spade, namely the Ace of Spades.

(b) 16 cards. There are 4 aces and there are 13 spades, but the Ace of Spades is in both of those groups. So, the total is $4 + 13 - 1$.

(c) 3 cards. There are 4 aces and only one of them is a spade, so there are 3 cards that are aces but are not spades.

(d) 49 cards. The only way for "Ace implies Spade" to be false is when "Ace" is true and "Spade" is false. So the only cards for which the statement is false are the three aces that are not spades. It is true for all of the other 52 minus 3 cards. Note that (d) is the negation of (c); that is, one is true if and only if the other is false. The answers for (c) and (d) have to add up to 52.

(e) 48 cards. "Ace implies King" is false when the card is an ace and is not a king, which is the case for all 4 aces. The statement "Ace implies King" is true for all of the 48 non-aces.

The answers could also be justified with a kind of truth table. For (a) through (d), we can divide the 52 cards into four categories, depending on whether "Ace" and "Spade" are true or false

Number of Cards	A	S	**(a)** $A \wedge S$	**(b)** $A \vee S$	**(c)** $A \wedge (\neg S)$	**(d)** $A \to S$
1	T	T	T	T	F	T
3	T	F	F	T	T	F
12	F	T	F	T	F	T
36	F	F	F	F	F	T

Then, for each part of the problem, just add up the numbers of cards corresponding to the T's in that column.